Homework 25: Line integrals

This homework is due Friday, 11/10 rsp Tuesday 11/14.

1 a) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = \langle x^3+y^3, y-z, z^2+1 \rangle$ and $\vec{r}(t) = \langle 11t^2, 11t^3, 11t \rangle$ with $0 \leq t \leq 3$.
b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = \langle 15z, 15y, -15x \rangle$ and $\vec{r}(t) = \langle 2t, \sin(t), \cos(t) \rangle$, $0 \leq t \leq \pi$.

Solution:
(a) The integral is

$11 \int_0^2 \langle t^6+t^9, -t+t^3, 1+t^2 \rangle \cdot \langle 2t, 3t^2, 1 \rangle dt = 11 \int_0^2 1+t^2+3*t^2*(-t+t^3)+2$

This evaluates to 495628980.
(b) The integral is

$$15 \int_0^\pi \langle \cos(t), \sin(t), -2t \rangle \cdot \langle 2, \cos(t), -\sin(t) \rangle dt =$$

$$= 15 \int_0^\pi (2\cos t + \sin t \cos t + 2t \sin t) dt = 30\pi \ .$$

2 An electric current I produces a magnetic field $\vec{B}$ whose flow lines are circles circling the wire. Let $C : \langle r\cos(t), r\sin(t), 0 \rangle$. Ampères law is $\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$, where μ_0 is a constant called permeability. Show that the magnitude $B(r) = |\vec{B}|$ of the magnetic field at a distance r from the center of the wire is $B = \frac{\mu_0 I}{2\pi r}$. Note that B is a scalar function and $\vec{B}$ is a vector field. Use first the information provided to find the vector field $\vec{B}$.

Solution:
The vector field has the form $\vec{B} = B(r)\langle -y/r, x/r\rangle$ where B depends only on the radius r. Above $\langle -y/r, x/r\rangle$ is the unit vector tangent to a circle of radius r. Integrating over a concentric circle of radius r, we get: $\int_{C(r)} \vec{B} \cdot d\vec{r} = 2\pi B(r)$. Solving for $B(r)$, we find that $B(r) = \mu_0 I/(2\pi r)$ as desired.

3 Determine from each of the following cases, whether $\vec{F}$ is conservative (a gradient field or not). If it is, find a function f such that $\vec{F} = \nabla f$.

a) $\vec{F}(x, y) = \langle y + 4x^3 + y^6, -x + 6x^4y^5\rangle$

b) $\vec{F}(x, y) = \langle x + 7e^x \sin(y), y^4 + 7e^x \cos(y)\rangle$

c) $\vec{F}(x, y, z) = \langle x + y, y + x, z^5 - \sin(z)\rangle$

d) $\vec{F}(x, y, z) = \langle x^5, z^5, y\rangle$

Solution:
Check $Q_x - P_y$ and integrate if necessary. a) no, b) yes, c) yes, d) no.

4 Evaluate the line integral $\int_C \langle 1 - ye^{-x}, e^{-x}\rangle \cdot d\vec{r}$, where C is the path $r(\vec{t}) = \langle t, 1 + t + \sin(\sin(t))\rangle$ and t is from 0 to π. You probably will have difficulty. A future "you" (who has a time machine) tells you that you can compute the integral also in a different way: find a function f which is a potential to the vector field, then evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0))$. You can use this without justification for now. We will learn about this "warp" feature in the next lecture.

Solution:
Check first that the integrand is conservative:

$$\frac{\partial(1 - ye^{-x})}{\partial y} = -e^{-x} = \frac{\partial(e^{-x})}{\partial x}.$$

The function $f = x+ye^{-x}$ satisfies $\nabla f = \langle 1-ye^{-x}, e^{-x}\rangle$, hence the answer should be $f(\pi, 1+\pi)-f(0,1) = e^{-\pi}+\pi+\pi e^{-\pi}-1$.

5 The topological notions appearing in this problem are cool but not very essential for the course. They play a role next week. Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.
a) $\{(x, y) \,|\, 1 < y < 3\}$, b) $\{(x, y) \,|\, 1 < |x| < 2\}$
c) $\{(x, y) \,|\, 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$ d) $\{(x, y) \,|\, (x, y) \neq (1, 2)\}$
e) $\{(x, y, z) \,|\, (x, y, z) \neq (1, 2, 3)\}$
f) $\{(x, y, z) \,|\, (x, y, z) \neq \{(\cos(t), \sin(t), 0) \mid 0 \leq t \leq 2\pi\}$

Solution:
a) The set is a horizontal strip which is open, connected and simply-connected.
b) The set is open but is not connected because it has two components, one where x is positive, and one where x is negative.
c) The set is a semi-annulus. It is connected and simply-connected; but since we have included the boundary, it is not open.
d) The set is a plane minus a point. It is open and connected; but since it has a hole at $(2, 3)$, it is not simply-connected.
e) The set **is** simply connected. You can pull together any string to a point.
f) The set is connected but not simply connected.

Main definitions

If $\vec{F}$ is a vector field and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ is called the **line integral** of $\vec{F}$ along the curve C. The field $\vec{F}$ is **conservative** in a region R if the line integral from A to B is path independent. It has the **closed loop property** if the line integral along any closed loop is zero. It is **irrotational** if $\text{curl}(F) = Q_x - P_y$ is zero everywhere in R.

A subset G of the plane is **open** if every point (x, y) in G is contained in a small disc D centered at (x, y) and $D \subset G$. (Openness is useful to make sure we do not hit a boundary where one has to worry about differentiability). G is **connected**, if one can connect any two points in G with a curve within G. it is **simply connected** if it is connected and every closed curve in G can be deformed to a point within G.

Clairaut test: Zero curl is necessary for a gradient field.