# Homework 26: Theorem of line integrals

This homework is due Monday, 11/13 rsp Tuesday 11/14.

1 a) Find a function $f$ such that $\vec{F} = \nabla f$ if $\vec{F}(x, y) = \langle x^8 - 2xy^2 + y, y^5 - 2x^2y + x\rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.

**Solution:**
(a) $f(x, y) = x^9/9 - x^2 * y^2 + y^6/6 + xy$.
(b) The integral equals $f(2, 8) - f(-1, 2) = 88155$

2 a) Find a function $f$ such that $\vec{F} = \nabla f$ if $\vec{F}(x, y) = \langle 30y^2/(1 + x^2), 60y \arctan(x)\rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $\vec{r}(t) = \langle t^2, 2t\rangle$ with $0 \leq t \leq 1$.

**Solution:**
(a) $f(x, y) = 30y^2 \arctan x$,
(b) The integral is $f(1, 2) - f(0, 0) = 30\pi - 0 = 30\pi$.

3 On August 1, 2017, Lukas Irmler walked over a rope over the **Rheinfalls** in Switzerland.

There is a force field $\vec{F}$ present which consists part of the gravitational force and part by the wind forces: $\vec{F}(x,y,z) = \langle \sin(x), \cos(y), -10+z \rangle$. The path is given by $\vec{r}(t) = \langle 5t, t, 30 - \sin(t)/10 \rangle$, where $0 \leq t \leq \pi$. Compute the work $\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ done by Lukas during this stunt.

**Solution:**
Best use the FTLI. The potential is $f(x,y,z) = -\cos(x) + \sin(y) - 10z + z^2/2$. Instead of integrating, we just have to evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0)) = f(5\pi, \pi, 30) - f(0,0,30) = 2$. It was also possible to do the line integral directly, but it was considerably more work.

4 a) Verify that if $\vec{F} = \langle P, Q, R \rangle$ is conservative, then

$$P_y = Q_x, P_z = R_x, Q_z = R_y \ .$$

b) Is $\langle x^5 y, xy^2, zx \rangle$ conservative? If yes, find $f$ such that $\vec{F} = \nabla f$, if not, give a reason.

**Solution:**
(a) The definition of a conservative vector field is one that is independent of the choice of path. By sticking to paths that lie in planes parallel to the coordinate planes, we get the result. For example, working in planes parallel to the $xy$-plane tells us that $P_y = Q_x$.
(b) No, since $P_y = x^5 \neq y^2 = Q_x$.

5 a) Show that the vector field $\vec{F}(x,y,z) = \langle y, x, xyz \rangle$ is not con-

servative by using problem 4).

b) Find two different curves $C_1, C_2$ from $(0,0,0)$ to $(1,1,0)$ for which the line integrals of $\vec{F}$ along $C_1, C_2$ are different.

**Solution:**
(a) For example, $P_z = 0 \neq yz = R_x$.
(b) Unfortunately, we cannot take both curves that lie entirely in the $xy$-plane since $P_y = 1 = Q_x$. Consider a vertical line segment $I_{x,y}$ which joins $(x,y,0)$ to $(x,y,1)$. Then the integral over this segment is $\int_0^1 xyz dz = xy/2$. This gives us a hint how to proceed. Take $\gamma_1$ to be any curve from $(0,0,0)$ to $(1,1,0)$ in the $xy$-plane. Set $\gamma_2$ be the union of $I_{0,0}$ (oriented upwards), a curve in the plane $z = 1$ above $\gamma_1$ and $I_{1,1}$ (oriented downwards). The difference between the integrals over $\gamma_2$ and $\gamma_1$ is $xy/2|_{1,1} - xy/2|_{0,0} = 1/2$ so the two integrals are not the same.

## Main points

This theorem is the first generalization of the fundamental theorem of calculus to higher dimensions. It tells that the work done along a path is the potential energy difference.

**Fundamental theorem of line integrals**: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

This theorem can be used to dramatically simplify the computation of a line integral. Just find the potential $f$ and evaluate the difference of potential values.

Recall that a region $R$ is called **simply connected** if every closed loop in $R$ can be pulled together to a point within $R$.

The three concepts "gradient field", "closed loop property" and "conservative" are the same:
Gradient field $\leftrightarrow$ Conservative $\leftrightarrow$ Closed loop property
In simply connected open regions, these three properties are all equivalent to being irrotational $\mathrm{curl}(\vec{F}) = Q_x - P_y = 0$.