

Homework 27: Green's theorem

This homework is due Wednesday, 11/15 resp Thursday 11/16.

1 Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} ,$$

where $\vec{F}(x, y) = \langle 123y + 3x^{15} - 1, 17x + \cos(9y)y^{300} \rangle$ and C consists of the line segments from $(0, 2)$ to $(0, 0)$ and from $(0, 0)$ to $(2, 0)$ and the curve $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(0, 2)$.

Solution:

We have $Q_x - P_y = 17 - 123 = -106$ so that the result is -106 times the area of a quarter circle with radius 2. The result is $-106\pi = -106\pi$.

2 a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ with

$$\vec{F}(x, y) = \langle y^2 \cos(x) + \sin(\sin(x)), 45x + x^3 + 2y \sin(x) + \sin(\sin(y)) \rangle$$

where C is the triangular path from $(0, 0)$ to $(2, 6)$ to $(2, 0)$ to $(0, 0)$. Watch the orientation of the curve!

b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x, y) = \langle -8y + 1/(1 + x + y^3), 7x + 3y^2/(1 + x + y^3) \rangle$$

and C is the circle $(x - 6)^2 + (y - 7)^2 = 49$ oriented counterclockwise.

Solution:

a) The curl is $45 + 3x^2$. We have to integrate this function over the triangle.

$$\int_0^2 \int_0^{3x} (45 + 3x^2) dy dx .$$

The result of this integration is 306. Since the orientation is opposite, the result is -306 .

b) The curl is 15. The result is the area of the disk times 15 which is $15 \cdot 49\pi = 735\pi$.

- 3 A classical problem asks to compute the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle 4 \cos^3(t), 4 \sin^3(t) \rangle, 0 \leq t \leq 2\pi .$$

We can not do that directly. Guess which theorem to use, then use it!

Solution:

Take a vector field $\vec{F}(x, y) = \langle 0, x \rangle$ which has the curl 1. Then by Green the area is the line integral

$$\begin{aligned} & \int_0^{2\pi} \langle 0, 4 \cos^3(t) \rangle \cdot \langle -12 \cos^2(t) \sin(t), 12 \sin^2(t) \cos(t) \rangle dt \\ &= 24 \int_0^{2\pi} (\cos^2(t) \sin^2(t)) (\cos^2(t)) dt \\ &= 24 \int_0^{2\pi} \frac{\sin^2(2t)}{4} \left(\frac{\cos(2t) + 1}{2} \right) \\ &\Rightarrow 6\pi. \end{aligned}$$

- 4 Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle x^2 + y, 3x - \sin(y^2) \rangle$ and C is the ellipse $x^2/100 + y^2/16 = 1$ oriented clockwise.

Solution:

P and Q have continuous partial derivatives on \mathbf{R}^2 , so by Green's Theorem we have 2 times the area 40π of the ellipse. But because the curve is oriented clockwise, the result is -80π .

5 Use Green's Theorem to evaluate

$$\int_C \langle \sin(\sqrt{1+x^3}), 7x \rangle d\vec{r},$$

where C is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length $1/3$ added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length $1/9$ added. $K(3)$ is $K(2)$ with $3 * 4^2$ of length $1/27$ added and $K(4)$ is $K(3)$ with $3 * 4^3$ triangles of length $1/81$ added. Remark. We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26\dots$ which is between 1 and 2.



Solution:

Since $\text{curl}(F) = 7$, we have to compute the area of $K(3)$ and multiply by 7. We have $|K(3)| = \left(\frac{\sqrt{3}}{4}\right) \left(1 + \frac{3}{9} + \frac{12}{81} + \frac{48}{729} + \frac{192}{(9 * 729)}\right) = 4.778$.

This was not required: in the limit we have a geometric series $|K| = \frac{\sqrt{3}}{4} \left(1 + \frac{3}{9} \sum_{k=0}^{\infty} \frac{4^k}{9^k}\right) = \frac{\sqrt{3}}{4} \left(1 + \frac{3}{9} \left(\frac{1}{1 - \frac{4}{9}}\right)\right) = \frac{2\sqrt{3}}{5}$.

Main points

The **curl** of a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is the scalar field

$$\text{curl}(F)(x, y) = Q_x(x, y) - P_y(x, y) .$$

Green's theorem: If $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a vector field and G is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_G \text{curl}(F) \, dx dy .$$