Homework 28: Curl and Div

This homework is due Friday, 11/17 rsp Tuesday 11/21.

1 a) Find the curl of the vector field

$$\vec{F}(x,y,z) = \langle \sin(y+z), \sin(z+x), \sin(x+y) \rangle \ .$$

b) Find the divergence of the gradient of $f(x,y,z) = x^3 + y^5 + 3z^2y$ c) Define $\Delta f = f_{xx} + f_{yy} + f_{zz}$. After having done a) and b), evaluate div(curl($\vec{F}$)), Δf at $(1,1,1)$ and div(grad(f) and curl(grad)(f) **in your head**! There is no need here to write down anything than the three numbers.

Solution:
a) $\langle \cos(x+y) - \cos(x+z), \cos(y+z) - \cos(x+y), \cos(x+z) - \cos(y+z) \rangle$.
b) $6x + 6y + 20y^3$.
c) 0, 32, 0.

2 Let f be a scalar field and $\vec{F}$ a vector field in space. Determine which expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field. In all problems, we deal with functions and vector fields in 3D space.

a)	curl(div($\vec{F}$))	b)	curl(grad(f))
c)	grad($\vec{F}$)	d)	grad(div($\vec{F}$)
e)	div(grad(f))	f)	grad(div(f))
g)	curl(curl($\vec{F}$))	h)	div(div($\vec{F}$))
i)	grad(f) × div($\vec{F}$)	j)	div(curl(grad$f$))
k)	curl(f)	l)	curl(div(f))

Solution:

(a) div$\vec{F}$ is a scalar field, the curl is not defined.

(b) curl(grad(f)) is a vector field.

(c) grad($\vec{F}$) is meaningless because $\vec{F}$ is not a scalar field.

(d) grad(div($\vec{F}$) is a vector field.

(e) div(grad(f)) is a scalar field.

(f) grad(div(f)) is meaningless because f is a scalar field.

(g) curl(curl($\vec{F}$)) is a vector field.

(h) div(div($\vec{F}$)) is meaningless because div$\vec{F}$ is a scalar field.

(i) grad(f) $\times$ div($\vec{F}$) is meaningless because div$\vec{F}$ is a scalar field.

(j) div(curl(grad$f$)) is a scalar field.

(k) curl$f = \nabla \times f$ is meaningless because f is a scalar field.

(l) The divergence of f is not defined.

3 a) Is there a vector field $\vec{G}(x, y, z)$ such that curl($\vec{G}$) $= \langle 8, 7, 12 \rangle$. If yes, find one.
b) Is there a vector field $\vec{G}(x, y, z)$ such that

$$\text{curl}(\vec{G}) = \langle xyz, -y^2 z, yz^2 \rangle \ ?$$

If yes, find one.
c) Assume $\vec{F}$ is a gradient field. Does this imply that there is a vector field $\vec{G}$ such that curl($\vec{G}$) $= \vec{F}$? If yes, show it. If no, find a counter example.

Solution:

a) Yes, for example $\langle -12y, -8z, -7x\rangle$. b) No. Assume there is such a G. Then $\text{div}(\text{curl}\vec{G}) = \frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(-y^2z)+\frac{\partial}{\partial z}(yz^2) = yz - 2yz + 2yz = yz \neq 0$ which contradicts Theorem 11.

c) We necessarily need the divergence to be zero. Take $\langle x, 0, 0\rangle$ for example.

4 a) Verify that any vector field of the form

$$\vec{F}(x, y, z) = \langle f(x), g(y), h(z)\rangle$$

is irrotational.

b) Verify that any vector field of the form

$$\vec{F}(x, y, z) = \langle f(y, z), g(x, z), h(x, y)\rangle$$

is incompressible.

c) Find a nonzero vector field $\vec{F}$ such that $\text{curl}(\vec{F}) = \langle 0, 0, 0\rangle$.

Solution:

(a) $\text{curl}\vec{F} = \left(\frac{\partial h(z)}{\partial y} - \frac{\partial g(y)}{\partial z}\right)\vec{i} + \left(\frac{\partial f(x)}{\partial z} - \frac{\partial h(z)}{\partial x}\right)\vec{j} + \left(\frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y}\right)\vec{k} = (0-0)\vec{i} + (0-0)\vec{j} + (0-0)\vec{k} = 0.$

Hence $\vec{F}$ is irrotational.

(b) $\text{div}\vec{F} = \frac{\partial(f(y,z))}{\partial x} + \frac{\partial(g(x,z))}{\partial y} + \frac{\partial(h(x,y))}{\partial z} = 0$ so $\vec{F}$ is incompressible.

(c) Take a gradient field.

5 a) Prove the identity $\text{div}(\nabla f \times \nabla g) = 0$.

b) Show that the scalar function $f(x,y,z) = 3\sin(z) + z^5 + y + x$ is the divergence of some vector field $\vec{F}$.

Solution:

a)Let ∇f be $\langle P_1, Q_1, R_1\rangle$ and $\nabla g = \langle P_2, Q_2, R_2\rangle$. Then

$$\mathrm{div}(\nabla f \times \nabla g) = \nabla \cdot (\nabla f \times \nabla g)$$

$$\begin{aligned}
&= \frac{\partial}{\partial x}(Q_1R_2 - R_1Q_2) - \frac{\partial}{\partial y}(P_1R_2 - R_1P_2) + \frac{\partial}{\partial z}(P_1Q_2 - Q_1P_2) \\
&= \left[Q_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial Q_1}{\partial x} - Q_2\frac{\partial R_1}{\partial x} - R_1\frac{\partial Q_2}{\partial x}\right] - \\
&\quad \left[P_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial P_1}{\partial y} - P_2\frac{\partial R_1}{\partial y} - R_1\frac{\partial P_2}{\partial y}\right] + \\
&\quad \left[P_1\frac{\partial Q_2}{\partial z} + Q_2\frac{\partial P_1}{\partial z} - P_2\frac{\partial Q_1}{\partial z} - Q_1\frac{\partial P_2}{\partial z}\right] \\
&= \left[P_2\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right] - \\
&\quad \left[P_1\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right] \\
&= \nabla g \cdot \mathrm{curl}(\nabla f) - \nabla f \cdot \mathrm{curl}(\nabla g) \\
&= 0 \text{ (by Theorem 3)}
\end{aligned}$$

b) It is easier to make the problem (seemingly) harder by assuming $P = 0, Q = 0$. Then $\mathrm{div}(F) = R_z$. The question is now whether any function can appear in this form. Yes, just take $R(x,y,z) = \int_0^z f(x,y,t)\ dt$. In our case it is $\vec{F} - \langle 0, 0, -3\cos(z) + z^6/6\rangle$.

Main points

The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ is the vector field $\text{curl}(P, Q, R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$.

The curl measure rotation of a field. If $\vec{F}$ has zero curl everywhere it is **irrotational**. Remember that in two dimensions, the curl of $\vec{F} = \langle P, Q \rangle$ is a scalar.

The **divergence** of a vector field $\vec{F}(x, y, z) = \langle P, Q, R \rangle$ is

$$\text{div}(\vec{F})(x, y, z) = P_x(x, y, z) + Q_y(x, y, z) + R_z(x, y, z).$$

$\text{div}(\vec{F})$ measures the expansion of a field. Zero divergence everywhere is called **incompressible**.

$\text{div}(\text{curl}(\vec{(F)})) = 0$ for all vector fields $\vec{F}$. $\text{curl}(\nabla f) = 0$ for all functions f.