Homework 29: Flux integral, Stokes I

This homework is due Monday, 11/20 rsp Tuesday 11/21 just before Thanksgiving. If no orientation is given for a surface, the orientation is assumed to be "outwards". There is one problem which gives a first exposure to Stokes theorem The Monday/Tuesday lecture before thanksgiving will cover the theorem again.

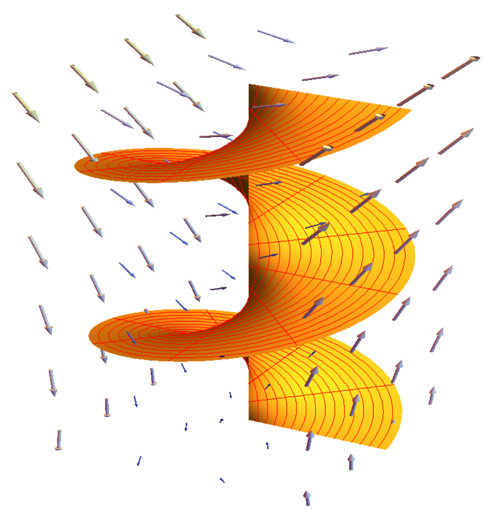
1 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ if

$$\vec{F}(x,y,z) = \langle 7x, y^2, z \rangle \ ,$$

and S is the helicoid

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, v \rangle, 0 \leq u \leq 1, 0 \leq v \leq 6\pi$$

which has an upward orientation.



Solution:
Compute $F(r(u,v))$ and $r_u \times r_v$ then take the dot product. We get then the integral $\int_0^1 \int_0^{6\pi} -u^2\sin^2(v)\cos(v) + v\left(u\sin^2(v) + u\cos^2(v)\right) + 7u\sin(v)\cos(v)\ dudv$. the result is $9\pi^2$.

2 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ for the vector field

$$\vec{F}(x,y,z) = \langle x, y, 5 \rangle \ ,$$

where S is part of the cylinder $x^2 + z^2 = 1$ bound by $y = 0$ and $y = 7$. The surface is oriented outwards.

Solution:
Parametrize the surface S: $\vec{F}(\vec{r}(\theta, y)) = \langle \sin\theta, y, 5 \rangle$ and $\vec{r}_\theta \times \vec{r}_y = \langle \sin\theta, 0, \cos\theta \rangle \Rightarrow$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^7 \left(\sin^2\theta + 5\cos\theta\right) \, dy \, d\theta = 7\pi \ .$$

3 The temperature $f(x, y, z)$ at a point (x, y, z) is equal to the distance from the origin $(0, 0, 0)$. Find the flux of the heat gradient field $\vec{F} = -\nabla f$ across a sphere S of radius 2 centered at $(0, 0, 0)$.

Solution:

$u(x,y,z) = \sqrt{x^2+y^2+z^2}$. Then,

$$\vec{F} = -\nabla u = \left\langle -\frac{x}{\sqrt{x^2+y^2+z^2}}, -\frac{y}{\sqrt{x^2+y^2+z^2}}, -\frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

The outward unit normal to the unit sphere of radius a is $\vec{n} = \langle x/a,\ y/a,\ z/a \rangle$. Therefore

$$\vec{F} \cdot \vec{n} = \frac{\langle -x, -y, -z \rangle \cdot \langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2}} = -1.$$

Thus, the rate of heat flow across S is

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \vec{n}\, dS \\
&= \iint_S -1\, dS \\
&= -(\text{Surface area of } S) \\
&= -4\pi r^2 \\
&= -16\pi.
\end{aligned}$$

4 Let $\vec{F}(x,y,z)$ be an inverse square field, that is

$$\vec{F}(x,y,z) = c\langle x, y, z \rangle / \rho^3$$

with $\rho = \sqrt{x^2+y^2+z^2}$. Show that the flux of $\vec{F}$ across a sphere S with center at the origin and radius R is independent of the radius of the sphere.

Solution:
Let $\vec{r} = \langle x, y, z\rangle$ and S be a sphere of radius ρ centered at the origin. Then $|\vec{r}| = \rho$ and $\vec{F}(\vec{r}) = c\vec{r}/|\vec{r}^3| = \left(\frac{c}{\rho^3}\right)\langle x, y, z\rangle$. A parametric representation for S is $\vec{r}(\phi, \theta) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\vec{r}_\phi = \langle \rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, -\rho\sin\phi\rangle$, $\vec{r}_\theta = -\rho\sin\phi\sin\theta, \rho\sin\phi\cos\theta, 0\rangle$ and the outward orientation is given by

$$\vec{r}_\phi \times \vec{r}_\theta = \langle \rho^2\sin^2\phi\cos\theta, \rho^2\sin^2\phi\sin\theta, \rho^2\sin\phi\cos\phi\rangle$$

The flux of $\vec{F}$ across S is

$$\begin{aligned}\iint_S \vec{F}\cdot d\vec{S} &= \int_0^\pi \int_0^{2\pi} \frac{c}{\rho^3}\langle \rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi\rangle \\ &\cdot\ \langle \rho^2\sin^2\phi\cos\theta, \rho^2\sin^2\phi\sin\theta, \rho^2\sin\phi\cos\phi\rangle\, d\theta\, d\phi \\ &= \frac{c}{\rho^3}\int_0^\pi \int_0^{2\pi} \rho^3(\sin^3\phi + \sin\phi\cos^2\phi)\, d\theta\, d\phi \\ &= c\int_0^\pi \int_0^{2\pi} \sin\phi\, d\theta\, d\phi \\ &= 4\pi c.\end{aligned}$$

Thus the flux does not depend on the radius ρ.

5 Use Stokes theorem to evaluate the flux integral $\int\int_S \mathrm{curl}(\vec{F})\cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle (x^9 + y^7)z^5, x, y\rangle\ ,$$

where S is the surface $x^2 + y^2/16 + z^8 = 1, z \ge 0$, oriented upwards. Stokes theorem expresses this as a line integral along the boundary curve $\vec{r}(t) = \langle \cos(t), 4\sin(t), 0\rangle$, $0 \le t \le 2\pi$.

Solution:
Compute the line integral of $\vec{F}$ along the circle of radius 5 in the $x-y$ plane. The result is 4π.

Main points

If a surface S is parametrized as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a domain G in the uv-plane and $\vec{F}$ is a vector field, then the **flux integral** of $\vec{F}$ through S is

$$\int\int_G \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv\ .$$

With $d\vec{S} = (\vec{r}_u \times \vec{r}_v)\ dudv$, this can be written as $\int\int_S \vec{F} \cdot d\vec{S}$. The interpretation is that if $\vec{F}$ = fluid velocity field, then $\int\int_S \vec{F} \cdot d\vec{S}$ is the amount of fluid passing through S in unit time.

Stokes theorem tells that if S be a surface bounded by a curve C and $\vec{F}$ be a vector field, then

$$\int\int_S \mathrm{curl}(\vec{F})\ \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}\ .$$