Homework 3: Cross product, lines, planes

This homework is due Wednesday, 9/13 rsp Thursday 9/14.

1 a) Find the equation of the plane containing the three points $P = (1, 1, 1), Q = (4, 6, 3), R = (5, 5, 0)$. b) Find the area of the triangle PQR.

Solution:
a) Two vectors in the plane through the points P, Q, and R are the vectors $\overrightarrow{PQ} = \langle 3, 5, 2\rangle$ and $\overrightarrow{PR} = \langle 4, 4, -1\rangle$. Thus, a vector orthogonal to the plane through P, Q, R is
$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -13, 11, -8\rangle$. The equation is $-13x + 11y - 8z = d\rangle$. We find d by by plugging in a point like $(x, y, z) = (1, 1, 1)$. The result is $-13x + 11y - 8z = -10$. b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle -13, 11, -8\rangle| = \sqrt{354}$. Thus, the area of triangle PQR is $\frac{1}{2}\sqrt{354}$.

2 a) Compute a suitable volume to determine whether $A = (2, 2, 3), B = (4, 0, 7), C = (6, 3, 1)$ and $D = (2, -3, 11)$ are in the same plane. b) Find the distance between the line L through A, B and the line M through C, D.

Solution:
a) Begin by defining the vectors $\vec{u} = \overrightarrow{AB} = \langle 2, -2, 4\rangle, \vec{v} = \overrightarrow{AC} = \langle 4, 1, -2\rangle$ and $\vec{w} = \overrightarrow{AD} = \langle 0, -5, 8\rangle$. These three vectors determine a parallelepiped whose volume can be computed by their triple scalar product. If the vectors are coplanar, then the volume of this parallelepiped will be 0. Compute the triple scalar product:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = 20$$

As the volume is not zero, the vectors are not coplanar b) Since the distance is the volume divided by the area $|\vec{u} \times \vec{v}| = |\langle 0, 20, 10\rangle|$. The volume is 20, the area of the base is $10\sqrt{5}$ which gives $2/\sqrt{5}$. But this is the distance from the point D to the plane. (The original solution here was computing this). To get the distance between the two lines, we have to divide by the area of the parallelogram spanned by ab and cd which is $4\sqrt{107}$. The result is then $5/\sqrt{107}$.

3 a) Find an equation of the plane containing the line of intersection of the planes $x - z = 1$ and $y + z = 3$ which is perpendicular to the plane $x + y - 2z = 1$.
b) Find the distance of the plane found in a) to the origin $(0, 0, 0)$.

Solution:

a) To find the equation of a plane, we need to find a point and two vectors (that are not multiples of one another) in the plane. Since our plane is perpendicular to $x + y - 2z = 1$, it must contain $\vec{d_1} = \langle 1, 1, -2\rangle$ as a tangent vector. To find another vector in our plane, we note that the direction vector of the line of intersection also lies in our plane. This direction vector lies in both the plane $x - z = 1$ and the plane $y + z = 3$, so it is perpendicular to both their normal vectors. So, we compute:

$$\vec{d_2} = \langle 1, 0, -1\rangle \times \langle 0, 1, 1\rangle = \langle 1, -1, 1\rangle.$$

Thus, the plane we are looking for has two tangent vectors: $\langle 1, 1, -2\rangle$ and $\langle 1, -1, 1\rangle$. To find the normal vector to our plane, we cross these two to get

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \langle 1, 1, -2\rangle \times \langle 1, -1, 1\rangle = \langle -1, -3, -2\rangle.$$

This gives us the equation

$$-x - 3y - 2z = d.$$

Since the point (1, 3, 0) lies in our plane (it lies on the line of intersection), we use it to find that $d = -10$. Thus the equation our plane is:
$x + 3y + 2z = 10$.

b) We use the formula for the distance from a point P to a plane Σ: $d(P, \Sigma) = \frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$. Here, $\vec{n}$ is the normal vector to the plane, so $\vec{n} = \langle 1, 3, 2\rangle$. We have $P = (0, 0, 0)$ as the point away from the plane and $Q = (1, 3, 0)$ as the point on the plane. Thus $\vec{PQ} = \langle 1, 3, 0\rangle$ and the distance from our plane to the origin is

$$\frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|} = \frac{|\langle 1, 3, 0\rangle \cdot \langle 1, 3, 2\rangle|}{|\langle 1, 3, 2\rangle} = \frac{10}{\sqrt{1 + 9 + 4}} = \frac{10}{\sqrt{14}}.$$

4 a) Parametrize the line L through $P = (2, 1, 2)$ that intersects the line $x = 1 + t, y = 1 - t, z = 2t$ perpendicularly.

b) Parametrize the y-axis.

c) What is the distance from this line L to the y-axis?

Solution:

a) The point $P = (1, 1, 0)$ is on the line. The vector $\vec{v} = \langle 1, -1, 2 \rangle$ is inside the line. If $X = (1 + t, 1 - t, 2t)$ is a general point on the line, then $\vec{PX} = \langle t - 1, -t, 2t - 2 \rangle$ must be perpendicular to $\langle 1, -1, 2 \rangle$. The dot product is $\vec{PX} \cdot \vec{v} = \langle t - 1 + t + 4t - 4 = 6t - 5 = 0$ gives $t = 5/6$ and the point $X = (7/6, 5/6, 2/6)$ and vector $\vec{PX} = \langle -1/6, -5/6, -1/3 \rangle$ which is parallel to $\langle 1, 5, 2 \rangle$. The equation of the line is $\vec{r}(t) = \langle 1, 1, 2 \rangle + t\langle 1, 5, 2 \rangle$.

b) The y-axis is parametrized by $\vec{r}(t) = \langle 0, y, 0 \rangle$.

c) Use the distance formula. The point $O = \langle 0, 0, 0 \rangle$ is on the second line. First compute the vector $\vec{n}$ perpendicular to the two lines: $\vec{n} = \langle 1, 5, 2 \rangle \times \langle 0, 1, 0 \rangle = \langle -2, 0, 1 \rangle$. The distance is $|\vec{PO} \cdot \vec{n}|/|\vec{n}| = |\langle 1, 1, 0 \rangle \cdot \langle -2, 0, 1 \rangle|/\sqrt{5}$. Which is $= 2/\sqrt{5}$.

5 To compute the distance between a plane $ax + by + cz + dw = e$ in four dimensional space and a point P, we can use the known formula $|\vec{PQ} \cdot \vec{n}|/|\vec{n}|$ from three dimensional space. Its just that the vector $\vec{n} = \langle a, b, c, d \rangle$ has now four coordinates and Q is a point on the plane. Find the distance between the plane $x + 3y + 5z + w = 1$ to the point $P = (1, 1, 1, 1)$.

Solution:

We have $\vec{n} = \langle 1, 3, 5, 1 \rangle$. Take $Q = (1, 0, 0, 0)$ for example. We have $\vec{PQ} = \langle 0, -1, -1, -1 \rangle$. We have $|\vec{PQ} \cdot \vec{n}| = 9$. The distance is $9/6$.

Main definitions

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in space is defined as the vector

$$\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle \ .$$

The number $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$. It satisfies $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The number $|[\vec{u}, \vec{v}, \vec{w}]|$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$. The **orientation** given by the sign of $[\vec{u}, \vec{v}, \vec{w}]$.

A point $P = (p, q, r)$ and a vector $\vec{v} = \langle a, b, c \rangle$ define the **line** $L = \{ \langle x, y, z \rangle = \langle p, q, r \rangle + t\langle a, b, c \rangle, t \in \mathbb{R} \}$.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{\vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.
An example is $\Sigma = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t\langle 2, 4, 6 \rangle + s\langle 1, 0, -1 \rangle \ \}$. This is called the **parametric description** of a plane. The implicit equation of the plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$ is $ax + by + cz = d$, where $\langle a, b, c \rangle = \vec{v} \times \vec{w}$ is a vector normal to the plane and d is obtained by plugging in $\vec{x}_0$. For distance formulas (which classes can not cover all) see the important handout on the website.