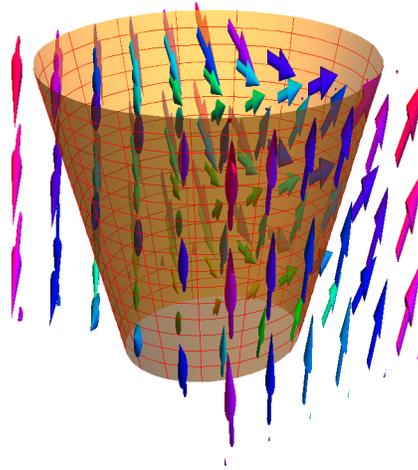## Homework 30: Stokes Theorem

This homework is due Wednesday, 11/29 or Tuesday 11/28 after thanksgiving.

1 Evaluate $\int \int_S \text{curl}(\mathbf{F}) \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle -3y, x, z + \sin(z - 1) \sin(z - 2) \rangle$, and where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. The cone can be parametrized by $\vec{r}(u, v) = \langle v \cos(u), v \sin(u), v \rangle$ with $0 \leq u \leq 2\pi$ and $1 \leq v \leq 2$.

Solution:

The boundary consists of two circles. At $z = 1$ we have the curve

$$\vec{r}(t) = \langle \cos(t), \sin(t), 1 \rangle$$

At $z = 2$ we have

$$\vec{r}(t) = \langle 2\cos(t), -2\sin(t), 2 \rangle.$$

The second curve is oriented clockwise so that the surface is to the left. By Stokes theorem, we can compute the flux $\int\int_S \text{curl}(\mathbf{F}) \cdot d\vec{S}$ is the sum of the two line integrals

$$\int_0^{2\pi} \langle -3\sin(t), \cos(t), 1 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle = dt = -2\pi$$

and

$$\int_0^{2\pi} \langle 6\sin(t), -2\cos(t), 2 \rangle \cdot \langle -2\sin(t), 2\cos(t), 0 \rangle \, dt = -16\pi$$

The answer is $\boxed{-18\pi}$.

2 Evaluate the line integral $\int_C \vec{F} \cdot d\mathbf{r}$, where $\vec{F}(x, y, z) = \langle x^3 - 4e^{-\cos(x)}, 3e^x, z^5 + e^z \rangle$ and where C is the boundary of the part of the plane $2x + y + z = 5$ in the first octant, oriented counterclockwise as viewed from above.

Solution:

Use Stokes theorem. We parametrize the surface as

$$\vec{r}(u, v) = \langle u, v, 5 - 2u - v \rangle .$$

We have $\vec{r_u} \times \vec{r_v} = \langle 2, 1, 1 \rangle$. The curl of the vector field is $\langle 0, , 3e^u \rangle$. Integrating gives

$$\int_0^{5/2} \int_0^{(5-u)/2} 3e^u \, dv du$$

The answer is $\int_0^{5/2} 15e^v - 6v \, dv = 6e^{5/2} - 21$.

3 Evaluate the line integral $\int_C \vec{F} \cdot d\mathbf{r}$, where $\vec{F}(x, y, z) = \langle xy, 2z, 3y \rangle$ and C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

Solution:

The intersection of the plane and the cylinder is an ellipse lying over the disk D given by $x^2 + y^2 \le 9$ in the xy-plane. We parametrize by $\vec{r}(x, y) = \langle x, y, 5 - x \rangle$ for (x, y) lying the ellipse D. The usual computation finds

$$\vec{r}_x \times \vec{r}_y = \langle 1, 0, 1 \rangle \text{ and } \operatorname{curl} \vec{F} = \langle 1, 0, -x \rangle.$$

(Notice that the upward orientation from our parametrization of S is compatible with the orientation of C.) Now we apply Stokes's Theorem, eventually switching to polar coordinates to compute the integral:

$$
\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_D \operatorname{curl} \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) \, dx \, dy \\
&= \iint_D \langle 1, 0, -x \rangle \cdot \langle 1, 0, 1 \rangle \, dx \, dy = \iint_D (1 - x) \, dx \, dy \\
&= \int_0^{2\pi} \int_0^3 (1 - r \cos \theta) \, r \, dr \, d\theta = \int_0^{2\pi} \left(\frac{9}{2} - 9 \cos \theta \right) d\theta \\
&= 9\pi.
\end{aligned}
$$

4 Compute both sides of Stokes' Theorem for $\vec{F}(x, y, z) = \langle -2yz, y, 3x \rangle$ and the surface S which is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upwards.

Solution:

The paraboloid intersects the plane $z = 1$ when $1 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$, so the boundary curve C is the circle $x^2 + y^2 = 4$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\vec{r}(t) = \langle 2\cos t, 2\sin t, 1 \rangle$, $0 \leq t \leq 2\pi$, and then $\vec{r}'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$. Thus $\vec{F}(\vec{r}(t)) = \langle -4\sin t, 2\sin t, 6\cos t \rangle$, $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle 8\sin^2 t, 4\sin t \cos t, 0 \rangle$ and

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (8\sin^2 t + 4\cos t \sin t)dt$$
$$= 8(\frac{1}{2}t - \frac{1}{4}\sin 2t) + 2\sin^2 t|_0^{2\pi}$$
$$= 8\pi$$

Now curl $\vec{F} = \langle 0, -3 - 2y, 2z \rangle$. The surface $r(x, y) = \langle x, y, 5 - x^2 - y^2 \rangle$ satisfies $r_x \times r_y = \langle 1, 0, -2x \rangle times \langle 0, 1, -2y \rangle$ for $z5 - x^2 - y^2$, we have

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \iint_D [-0 - (-3 - 2y)(-2y) + 2z] \, dA$$
$$= \iint_D [-6y - 4y^2 + 2(5 - x^2 - y^2] \, dA$$
$$= \int_0^{2\pi} \int_0^2 [-6r\sin\theta - 4r^2\sin^2\theta + 2(5 - r^2)]r \, dr \, d\theta$$
$$= \left[-2r^3\sin\theta - r^4\sin^2\theta + 5r^2 - \frac{1}{2}r^4 \right]_{r=0}^{r=2} d\theta$$
$$= \int_0^{2\pi} (-16\sin\theta - 16\sin^2\theta + 20 - 8)d\theta$$
$$= 16\cos\theta - 16\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right) + 12\theta|_0^{2\pi}$$
$$= 8\pi$$

5 a) Evaluate $\int_C \vec{F}(x, y, z) \cdot d\vec{r}$ with $\vec{F}(x, y, z) = \langle y + \sin x, z^2 + \cos y, x^3 \rangle$, where C is the curve $\vec{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leq t \leq 2\pi$ which as you can see lies on the surface $z = 2xy$.

b) Explain without doing any computation that if S is the torus $\vec{r}(u, v) = \langle (2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v) \rangle$ with $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$ and $\vec{F}$ is a vector field like $\vec{F}(x, y, z) = \langle e^{e^{e^x}}, \sin\sin(y + z + x), x^{100} \rangle$ then $\iint_S \mathrm{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Solution:

(a) The curl is $\vec{F} = \langle -2z, -3x^2, -1 \rangle$. Since $\sin 2t = 2\sin t \cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C. Then the projection of S onto the xy-plane is the unit disk $D[x^2 + y^2 \leq 1]$. C is traversed clockwise (when viewed from above so S is oriented downward. Using Equation 13.6.10 with $g(x,y) = 2xy$, $P = -2z = -4xy, Q = -3x^2, R = -1$ and multiplying by -1 for the downward orientation, we have

$$
\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= -\iint_S \text{curl}\vec{F} \cdot d\vec{S} = -\iint_D \left[-(-4xy)(2y) - (-3x^2)(2x) \right. \\
&= -\iint_D (8xy^2 + 6x^3 - 1)dA \\
&= -\int_0^{2\pi} \int_0^1 (8r^3 \cos\theta \sin^2\theta + 6r^3 \cos^3\theta - 1)r\,dr\,d\theta \\
&= -\int_0^{2\pi} \left(\frac{8}{5}\cos\theta \sin^2\theta + \frac{6}{5}\cos^3\theta - \frac{1}{2} \right) d\theta \\
&= -\left[\frac{8}{15}\sin^3\theta + \frac{6}{5}(\sin\theta - \frac{1}{3}\sin^3\theta) - \frac{1}{2}\sin^3\theta) - \frac{1}{2}\theta \right]_0^{2\pi} \\
&= \pi
\end{aligned}
$$

6 Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S. Then $\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \iint_{H_1} \text{curl}\vec{F} \cdot d\vec{S} + \iint_{H_2} \text{curl}\vec{F} \cdot d\vec{S} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2} \vec{F} \cdot d\vec{r} = -\oint_{C_1} \vec{F} \cdot d\vec{r}$ so $\iint_S \text{curl}\vec{F} \cdot d\vec{S} = 0$.

Main points

Stokes theorem: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \,.$$

The orientation of S is given by the parametrization: the orientation of C is such that if you walk along C with the head in the "up" direction $\vec{r}_u \times \vec{r}_v$ and your nose into the $\vec{r}\,'$ direction, then your left foot is on the surface.

Written out in detail, we have

$$\int\int_R \text{curl}(\vec{F}(\vec{r}(u,v)))\cdot(\vec{r}_u\times\vec{r}_v)\ du dv = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\ dt$$

A lot of things come together here: surfaces, curves, dot product, cross product, triple scalar product, vector fields, double integrals and curl. What does it mean? From a SMBC cartoon: **"Stokes theorem? Yeah, thats how if you draw a loop around something, you can tell how much swirly is in it."**