Homework 31: Divergence Theorem

This last homework is due Friday, 12/1 rsp Thursday 11/30 in the last lecture.

1 Find the flux of the field $\vec{F}(x,y,z) = \langle x^3, y^3 - 3yz^2, z^3 \rangle$ through the boundary of the solid bounded by paraboloid $z = 16 - x^2 - y^2$ and the xy-plane. by using the divergence theorem.

Solution:
$\text{div}\vec{F} = 4x + 2x + 2 = 6x + 8$. Use cylindrical coordinates to compute

$$\iiint_E \text{div}(\vec{F}) dV = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} 3r^2) r dz dr d\theta \ .$$

which simplifies to 2048π.

2 Find the flux of the vector field $\vec{F}(x,y,z) = \langle x^2y + \cos^6(y), xy^2, 2xyz + e^{\sin(x)} \rangle$ through the outwards oriented solid bound by $x = 0, y = 0, z = 0$, and $x + 2y + z = 2$.

Solution:

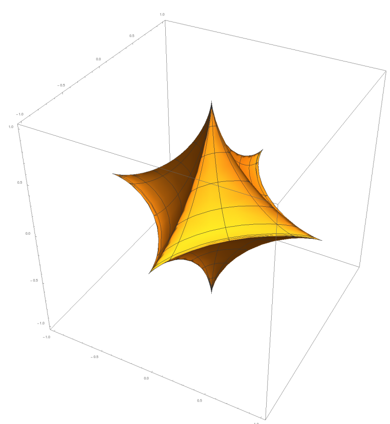
The divergence of $\vec{F}$ is

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(2xyz) = 2xy + 2xy + 2xy = 6xy.$$

Divergence Theorem turns the flux integral into a triple integral. Thus we get

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \iiint_E 6xy\, dV = \int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} 6xy\, dz\, dx\, dy \\
&= \int_0^1 \int_0^{2-2y} 6xy(2-x-2y) dx\, dy \\
&= \int_0^1 \int_0^{2-2y} (12xy - 6xy^2 - 12xy^2)\, dx\, dy \\
&= \int_0^1 \left[6x^2y - 2x^3y - 6x^2y^2\right]_{x=0}^{2-2y} dy \\
&= \int_0^1 y(2-2y)^3\, dy = \left[-\frac{8}{5}y^5 + 6y^4 - 8y^3 + 4y^2\right]_0^1 \\
&= \frac{2}{5}.
\end{aligned}$$

3 Use the divergence theorem to compute the volume of the solid enclosed by the surface parametrized by $\vec{r}(u,v) = \langle \cos^3(v)\sin^3(u), \sin^3(v)\sin^3(u), \cos^3(u)\rangle$ with $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

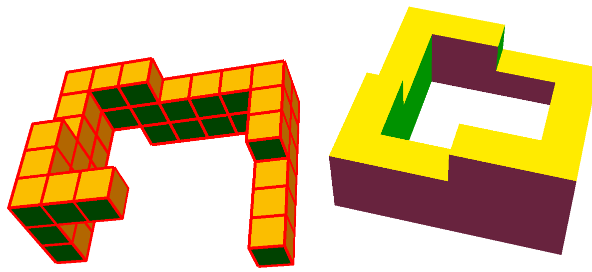


Solution:

Use $F = \langle 0, 0, z\rangle$ and find the flux of this vector field through the boundary. The result is $(4\pi/35)$.

4 Find $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle -x + \sin(y) + e^z, 30y +$

$\sin(z) + e^z, z + \sin(x) + e^y\rangle$ and S is the boundary of the Escher stair solid displayed in the picture. The right picture shows the same figure from an other angle leading to the illusion. Each brick is a cube of unit length 1.



Solution:

a) By the Divergence Theorem, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} dV = 30$ (volume of E) $= 37 * 30 = 1110$.

5 a) Use an integral theorem to evaluate $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle x^2yz, yz^2, z^3e^{xy}\rangle$, where is the part of upwards oriented surface $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$.
b) Use an integral theorem to compute the line integral of $\vec{F}(x, y, z) = \langle x^3, y^5, 2z\rangle$ along the path $\vec{r}(t) = \langle \cos(t) + t^{100} \sin(17t), \sin(t) + \sin(20t), t\rangle$ from $t = 0$ to $t = 10\pi$.

Solution:

a) $\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$ where C:

$$\begin{aligned} \vec{r}(t) &= \langle 2\cos t, 2\sin t, 1\rangle, 0 \le t \le 2\pi \\ \vec{r}'(t) &= \langle -2\sin t, 2\cos t, 0\rangle \\ \vec{F}(\vec{r}(t)) &= \langle 8\cos^2 t \sin t, 2\sin t, e^{4\cos t \sin t}\rangle \\ \text{and } \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= -16\cos^2 t\sin^2 t + 4\sin t\cos t \end{aligned}$$

Thus $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-16\cos^2 t \sin^2 t + 4\sin t \cos t) dt = \left[-16\left(-\frac{1}{4}\sin t\cos^3 t + \frac{1}{16}\sin 2t + \frac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi$

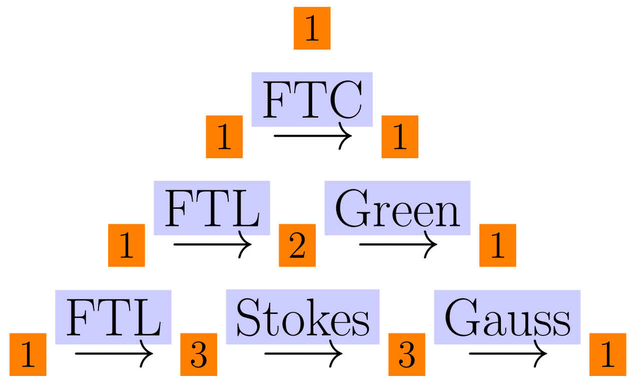
b) This is a gradient field with potential $x^4/4 + y^5/5 + z^2$. The initial point is $(1, 0, 0)$. The end point is $(1, 0, 10\pi)$. The result is $100\pi^2$.

Main points

Divergence Theorem. $\int\int\int_E \text{div}(\vec{F})\ dV = \int\int_S \vec{F} \cdot d\vec{S}$ All integral theorems are incarnations of **the fundamental theorem of multivariable Calculus**

$$\int_G dF = \int_{\delta G} F$$

where dF is a **derivative** of F and δG is the **boundary** of G.


1
FTC
1 1
FTL Green
1 2 1
FTL Stokes Gauss
1 3 3 1