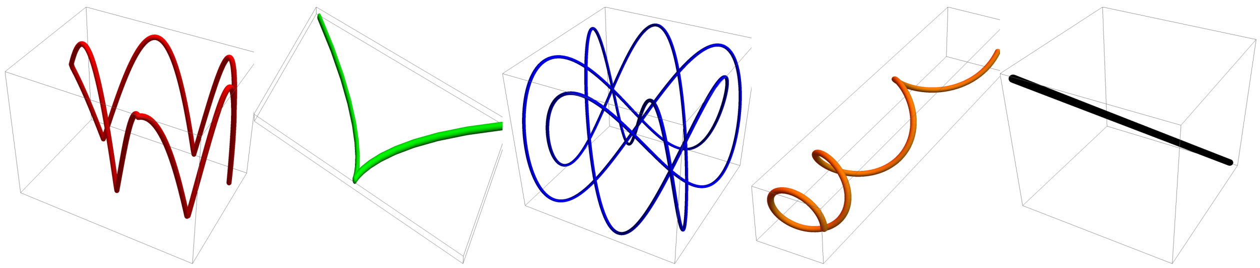
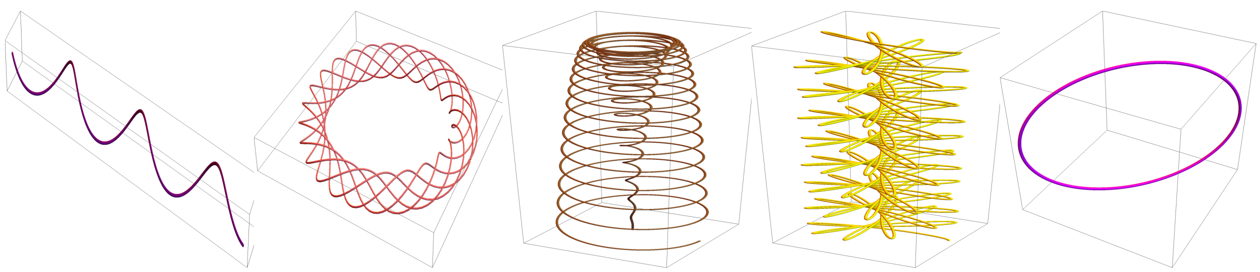
Homework 5: Parametrized curves

This homework is due Monday, 9/18 rsp Tuesday 9/19.

1 Match the curves:



I II III IV V



VI VII VIII IX X

$\vec{r}(t) =$	I-X
$\langle (6+\cos(24t))\cos(5t), (6+\cos(24t))\sin(5t), \sin(24t)\rangle$	
$\langle t\cos(8t), t\sin(8t), t(8\pi - t)\rangle$	
$\langle \cos(40t)\cos(3t), \cos(t40)\sin(4t), t\rangle$	
$\langle \cos(t), t^2, \sin(t)\rangle$	
$\langle t, \cos(t), \sin(t)\rangle$	
$\langle t^3, t^2, 0\rangle$	
$\langle \cos(3t), \sin(4t), \cos(7t)\rangle$	
$\langle \cos(t), \cos(t), \sin(t)\rangle$	
$\langle \lvert\cos(t)+\sin(t)\rvert, \lvert\sin(t)\rvert, \lvert\cos(5t)\rvert\rangle$	
$\langle t, t, -t\rangle$	

Solution:
$VII, VIII, IX, IV, VI, II, III, X, I, V.$

2 Parametrize the intersection of the wave surface $z = \sin(x)+\sin(y)$ with the elliptic cylinder $(x+5)^2/9 + (y-1)^2/16 = 1$.

Solution:
Start with the second equation which gives $x(t) = -5 + 3\cos(t), y(t) = 1 + 4\sin(t)$. Now plug into the third

$$\vec{r}(t) = \langle -5+3\cos(t), 1+4\sin(t), 3(-5+\sin(-5+3\cos(t)+\sin(1+4\sin(t)$$

for $t \in R$.

3 a) Two particles travel along the space curves $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$. Do the particles collide? Do the particle paths intersect?
b) If $\vec{r}(t) = \langle \cos(t), 2\sin(t), 4t \rangle$, find $\vec{r}\,'(0)$ and $\vec{r}\,''(0)$. Then compute $|\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$. We will later call this the curvature.

Solution:

(a) If they collide, the two particles must be at the same position at the same time. This gives us three equations: $t = 1 + 2t$, $t^2 = 1 + 6t$, and $t^3 = 1 + 14t$. Solving the first one gives $t = -1$. However, plugging this value of t into the second equation is not a solution, as $1 \neq 7$. Therefore, there is no collision.

If they intersect, the two particles must be at the same position at different times. This gives us the three equations: $t = 1 + 2s$, $t^2 = 1 + 6s$ and $t^3 = 1 + 14s$. We can solve them to get $s = 0, t = 1$ or $s = \frac{1}{2}, t = 2$. Taking the first solution, when $s = 0$ and $t = 1$, this gives us a path intersection at the point $(1, 1, 1)$. Taking the second solution, when $s = 0$ and $t = 1$, this gives us a path intersection at the point $2, 4, 8$.

(b) We compute:

$$\vec{r}\,'(0) = \langle -\sin(0), 2\cos(0), 4\rangle = \langle 0, 2, 4\rangle$$

$$\vec{r}\,''(0) = \langle -\cos(0), -2sin(0), 0\rangle = \langle -1, 0, 0\rangle.$$

Next, we compute the curvature at 0:

$$\frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3} = \frac{|\langle 0, -4, 2\rangle|}{\sqrt{20}^3} = \frac{\sqrt{20}}{\sqrt{20}^3} = \frac{1}{20}.$$

4 Find the point of intersection of two tangent lines to the curve $\vec{r}(t) = \langle \sin(\pi t), 2\sin(\pi t), \cos(\pi t)\rangle$. The first tangent is at $t = 0$, the second at $t = 0.5$.

Solution:
We parametrize the two lines as

$$\langle 0, 0, 1\rangle + t\langle \pi, 2\pi, 0\rangle$$

and

$$\langle 1, 2, 0\rangle + s\langle 0, 0, -\pi\rangle \ .$$

To find the point of intersection, we solve for t and s such that $\pi t = 1$, $2\pi t = 2$, and $1 = -\pi s$. We solve this to get that the paths intersect when $s = -1/\pi$ and $t = 1/\pi$ at the point (1,2,1).

5 A particle moving along a curve $\vec{r}(t)$ has the property that $\vec{r}\,''(t) = \langle 5, 6t, 8 + 4\sin(2t)\rangle$. We know $\vec{r}(0) = \langle 8, 1, 2\rangle$ and $\vec{r}\,'(0) = \langle 7, 0, 3\rangle$. What is $\vec{r}(\pi)$?

Solution:
Integrate $\vec{r}''(t)$ twice and fix the constants. The result is

$$\vec{r}(t) = \langle 8 + 7t + 5t^2/2, 1 + t^3, 2 + 5t + 4t^2 - \sin(2t)\ \rangle.$$

At $t = \pi$ we have $\langle 8 + 7\pi + 5\pi^2/2, 1 + \pi^3, 5\pi + 4\pi^2 rangle.$

Main definitions

The parametrization of a space curve is $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. The **image** of r is a **parametrized curve** in space. If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a curve, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t)\rangle$ is called the **velocity** at time t. Its length $|\vec{r}\,'(t)|$ is called **speed** and $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$ is called **unit tangent vector** or direction of motion. The vector $\vec{r}\,''(t)$ is called the **acceleration**. When knowing the acceleration and $\vec{r}\,'(0)$ and $\vec{r}(0)$ we can get back position $\vec{r}(t)$ by integration. Similarly, if we know $\vec{r}\,''(t)$ at all times and $\vec{r}(0)$ and $\vec{r}\,'(0)$, we can compute $\vec{r}(t)$ by integration.