Homework 6: Arc length and curvature

This homework is due Wednesday, 9/20 rsp Thursday 9/21.

1 a) Find the arc length of the curve $\vec{r}(t) = \langle 3\cos(t^5), 3\sin(t^5), 3t^5\rangle$ where $-1 \leq t \leq 1$.
b) Find the arc length of the curve $\vec{r}(t) = \langle t^3, 24t, 6t^2\rangle$, where $t \in [0, 6]$.

Solution:
a) We have $\vec{r}\ '(t) = 15t^4\langle\sin(t^4), \cos(t^4), 1\rangle$ and $|\vec{r}\ '(t)| = 15\sqrt{2}t^4$. Integrate to get $6\sqrt{2}$.
b) We have $|\vec{r}(t)| = 3(8 + t^2)$. Integrating this over the interval $[0, 6]$ gives 360.

2 Arc length can be defined in any dimensions. A curve in 4 dimension is parametrized as $\vec{r}(t) = \langle x_1(t), x_2(t), x_3(t), x_4(t)\rangle$. Find the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), 1/t, \log(t)\rangle \ ,$$

where $\log(t) = \ln(t)$ is the natural log and $1 \leq t \leq 4$.

Solution:
The velocity is $\vec{r}\ '(t) = \langle 1, 1/t, -1/t^2, 1/t\rangle$. Its length is $\sqrt{1 - 2/t^2 + 1/t^4}$ which factors. We have to integrate $(1+t^2)/t^2$ from $t = 1$ to $t = 4$. The answer is $15/4$.

3 a) Use a calculator, Mathematica or Wolfram alpha to evaluate the arc length of the curve $\vec{r}(t) = \langle Cos[t], Sin[t], t^3 \rangle$ from $t = 0$ to $t = 4$.
b) Do the same with $\vec{r}(t) = \langle Cos[t^2], Sin[t^2], t^6 \rangle$ from $t = 0$ to $t = 2$. Compare with the result in a) and explain you got the same result.

Solution:
a) The arc length equals to $\int_0^2 \sqrt{1 + 9t^6} dt$. Numerical integration gives us ~ 64.672.
b) the result is the same because we have a different parametrization. Take the time $s = t^2$ in the curve of a).

4 a) Use the formula

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

to compute the curvature $\kappa(t)$ of $\vec{r}(t) = \langle t^3, t^2, t \rangle$ at $t = 1$.
b) Find the curvature at $t = 1$ for the curve $\vec{r}(t) = 3\langle t^3, t^2, t \rangle$.
Hint. There is an easy way to see the answer.

Solution:
a) Differentiating, we find: $\vec{r}\ '(t) = \langle 3t^2, 2t, 1 \rangle$ and $\vec{r}\ ''(t) = \langle 6t, 2, 0 \rangle$. At $t = 1$, $\vec{r}\ '(1) = \langle 1, 2, 3 \rangle$, while $\vec{r}\ ''(1) = \langle 0, 2, 6 \rangle$. The cross-product is $\vec{r}\ '(1) \times \vec{r}\ ''(1) = \langle ..... \rangle$. This is $\sqrt{36 + 36 + 4}/\sqrt{14}^3 = 2\sqrt{19}/\sqrt{14}^3$.
b) The curve is scaled by a factor 3. The curvature therefore gets scaled by a factor $1/3$.

5 Find the parameter c such that the parabola $y = cx^2$ has curvature 40 at the origin.

Solution:

The curvature of a plane curve is given by

$$\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

Letting $t = x$ be the parameter, we find that

$$\kappa = \frac{2c}{(1 + 4c^2x^2)^{3/2}}.$$

At $t = x = 0$, we see that $\kappa = 2c$ so for κ to be 40, we need to take the parameter $c = 20$. We also can have $c = -20$ but this is the same parabola just flipped around.

Main definitions

If $t \in [a, b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

is called the **arc length of the curve**. Written out in coordinates, $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, we have

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt\ .$$

For curves in two dimensions, where $\vec{r}(t) = \langle x(t), y(t)\rangle$ has two coordinates only, we have $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**.

The **curvature** if a curve at the point $\vec{r}(t)$ is defined as

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} \; .$$

The curvature of a circle of radius r is equal to $1/r$ at every point of the circle. The curvature is zero for a line.

A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \; .$$