Homework 9: Functions and Continuity

This homework is due Friday, 9/29 rsp Tuesday 10/3.

1 Determine which f extends to a continuous function in the entire plane. No reasoning is required. Each question one point. $\log = \ln$ denotes the natural log.

a) $f(x,y) = xy(x^2-4)/(x+2)$.
b) $f(x,y) = \log(1+|xy|)$
c) $f(x,y) = (x^2-y^2)/x^2+y^2)$
d) $f(x,y) = \sin(x^2+y^2)$
e) $f(x,y) = 1/\log(2+|x+y|)$
f) $f(x,y) = y\exp(1/x)$
g) $f(x,y) = \log(\exp(x+y))$
h) $f(x,y) = \sin(ye^{\cos(x)})$
i) $f(x,y) = \sin(y)\exp(x)$
j) $f(x,y) = \exp(\log|x+y|)$

Solution:
a) Continuous (continue domain of definition to x=3)
b) Always continous
c) Not continuous at 0 d) Continuous everywhere
e) Continuous everywhere
f) Not continuous at $x = 0$
g) Continuous everywhere as it is x+y
h) Continuous
i) Continuous
j) Continuous. It is $|x+y|$.

2 Investigate whether the following functions are continuous at $(0,0)$. This means: investigate whether a function value at $(0,0)$ exists which extends the definition, so that the extended function is continuous. If the limit $(x,y) \to (0,0)$ exists, tell what the limit is.
a) $f(x,y) = x^2y/(x^2+y^2)^2$. b) $f(x,y) = xy/(x^2+y^2)$. It can be

helpful to use polar coordinates. You have to give some reasoning here.

Solution:
a) Not continuous. We have a factor r in the denomiator. b) Not continuous Use polar coordinates to see that it is $\sin(\theta)\cos(\theta)$.

3 Find the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists or show that the limit does not exist
a) $f(x,y) = \frac{6x^4y}{2x^5+y^5}$ b) $f(x,y) = \frac{x^6-y^6}{(x^2+y^2)^2}$
Also here is some reasoning required.

Solution:

a) The function is not continuous. In polar coordinates, it is a function of θ only. So, it depends from which direction we come to the origin.

b) The function is continuous. We can assign the value 0 to $(0,0)$. Claim: as $(x,y) \to (0,0)$,

$$\lim \frac{x^6}{(x^2+y^2)^2} \to 0.$$

Indeed,

$$|\frac{x^6}{(x^2+y^2)^2}| \leq |\frac{x^6}{(x^2)^2}| \leq x^2.$$

Since, $x^2 \to 0$ as $x \to 0$, this proves the claim. By the same reason,

$$\lim \frac{y^6}{(x^2+y^2)^2} \to 0.$$

It follows that the difference

$$\lim \frac{x^6-y^6}{(x^2+y^2)^2} \to 0.$$

4 Determine the set of points where the following function is continuous $f(x,y) = \frac{e^x+e^y}{e^{x+y}-1}$.

Solution:
The numerator and the denominator are continuous functions of x and y; hence the only problems in continuity can arise when the denominator equals to 0. This occurs when $e^{x+y} = 1$ or $x + y = 0$. The numerator is always positive, so we never have to worry about whether the limit of 0/0 exists. Hence, $H(x, y)$ is continuous at all points in the plane except where $x + y = 0$.

5 Find the limit $(x, y) \to (0, 0)$ of the function $f(x, y) = \sin(x^2 + y^2)\log(x^2 + y^2)$, where $\log(x) = \ln(x)$.

Solution:
First notice that as $(x, y) \to 0$, $x^2 + y^2 \to 0$. Setting $x^2 + y^2 = r^2$, it suffices to find the limit as $r \to 0$ of $\sin r \cdot \log r$. As $r \to 0$, $\sin r \to 0$ and $\log r \to -\infty$; so we have to be a bit careful because we are dealing with $0 \cdot (-\infty)$. One can use l'Hôpital's rule: write $u = r^2$ and take the limit

$$\frac{\log(u)}{(1/\sin(u))}$$

for $u \to 0$. By differentiating top and bottom we get the limit

$$(1/u)/(-\cos(u)/\sin^2(u)) = -(\sin(u)/u)(\sin(u)/\cos(u))$$

The first term goes to 1 by l'Hopital. The second goes to zero. The limit is zero.

Main definition

A function $f(x, y)$ with domain R is **continuous at** $(a, b) \in R$ if $f(x, y) \to f(a, b)$ for all choices $(x, y) \to (a, b)$. We also say that f **continuous at a point** (a, b) **not in the domain** if there exists a finite value $f(a, b)$ such that $f(x, y) \to f(a, b)$ whenever $(x, y) \to (a, b)$ for $(x, y) \in R$. For example, the function $f(x, y) = y(x^2 - 1)/(x - 1)$ continuous everywhere even so $x = 1$ is not in the domain. We can fill in the value $f(1, y) = 2y$ as the function is equivalent to $y(x+1)$, its analytic continuation. Also $f(x, y) = \sin(x^2 + y^2)/(x^2 + y^2)$ is continuous as l'Hopital for polar coordinates shows: with filling in the whole $f(0, 0) = 1$, it becomes continuous even at the point $(0, 0)$. In one dimensions, there are **jump discontinuities** like $f(x) = \text{sign}(x)$ or **poles** like $f(x) = 1/x$ or **oscillations** like $f(x) = \sin(1/x)$. These three prototypes can happen in the same function like in $1/\sin(1/x))$ or $\arctan(1/\sin(1/x))$. Many questions about continuity in two dimensions are answered when writing the function in polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ near the point in question. For $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ for example, the function becomes (just fill in $x = r\cos(\theta), y = r\sin(\theta)$), in polar coordinates $f(r, \theta) = \cos(2\theta)$. The value of the function depends only on the angle. Arbitrarily close to $(0, 0)$, the function takes any values between -1 and 1. The function is not continuous because no value can be found at $(0, 0)$ such that f can be continuously extended to it.