NOTES on FLUX INTEGRALS

Motivation: We already know how to compute the flux of a 2-dimensional vector field $\overrightarrow{\mathbf{F}}$ across a curve C in $\mathbf{R}^2$. For example, if $\overrightarrow{\mathbf{F}}$ is the velocity vector field of a fluid, then the flux of $\overrightarrow{\mathbf{F}}$ across C is the net rate of flow of the fluid across C (i.e., normal to C) and is given by

$$\text{Flux of } \overrightarrow{\mathbf{F}} \text{ across } C = \int_C \overrightarrow{\mathbf{F}} \cdot \vec{n}\, ds$$

where ds corresponds to Δs, a tiny arclength along the curve C, and $\vec{n}$ a unit normal.

Now we want to extend the concept of flux to three-space. In $\mathbf{R}^3$ we cannot define flux across a curve. (Our notion of "across" can no longer be well-defined since at any point on a curve in $\mathbf{R}^3$, there is a whole plane of normal lines to choose from as opposed to the unique normal line in $\mathbf{R}^2$.) We can, however, define the flux of $\overrightarrow{\mathbf{F}}$ across a surface S, where the notion of 'across' shall correspond to the direction normal to S, which is unique up to a sign. Basically, we'll think of chopping the surface S into many minuscule pieces (called surface elements) and summing the normal component of $\overrightarrow{\mathbf{F}}$ on each piece times the area of the piece. That is, we'll integrate $\overrightarrow{\mathbf{F}} \cdot \vec{\mathbf{n}}$ over the surface. We make the definition

$$\text{Flux of } \overrightarrow{\mathbf{F}} \text{ across } S = \iint_S \overrightarrow{\mathbf{F}} \cdot \vec{n}\, d\sigma$$

where $d\sigma$ corresponds to $\Delta\sigma$, the area of a tiny piece of the surface S. We need to understand what this means and how to compute it.

First let's look at some simple cases. Consider a fishnet S in flowing water. Let's think of $\overrightarrow{\mathbf{F}}$ as the velocity vector field for the water.

Case 1: $\overrightarrow{\mathbf{F}}$ is a constant vector field, $\overrightarrow{\mathbf{F}}(x, y, z) = \vec{\mathbf{v}}_0$, and S is a flat surface perpendicular to $\vec{\mathbf{v}}_0$.

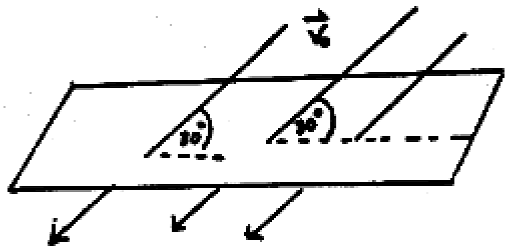
$$\text{Flux of } \overrightarrow{\mathbf{F}} \text{ across } S = \pm|\vec{\mathbf{v}}_0| \text{ (area of } S)$$

where $|\vec{\mathbf{v}}_0|$ = the speed at which the fluid moves, $\pm$ determined by orientation.

Case 2: $\overrightarrow{\mathbf{F}}$ is a constant vector field, $\overrightarrow{\mathbf{F}}(x, y, z) = \vec{\mathbf{v}}_0$, and S is a flat surface but *not* perpendicular to $\vec{\mathbf{v}}_0$.

$$\text{Flux of } \overrightarrow{\mathbf{F}} \text{ across } S = \pm(\vec{\mathbf{v}}_0 \cdot \vec{n})(\text{area of } S)$$

where $\vec{\mathbf{v}}_0 \cdot \vec{\mathbf{n}}$ is the component of $\overrightarrow{\mathbf{F}}$ in the direction perpendicular to S. For instance, if water is flowing along the surface, as opposed to through it, $\vec{\mathbf{v}}_0 \cdot \vec{\mathbf{n}}$ will be zero. If the water flow is as shown below, the flux is $(\vec{\mathbf{v}}_0 \cdot \vec{\mathbf{n}})(\text{area of } S) = \left(\frac{1}{2}|\vec{\mathbf{v}}|\right)(\text{area of } S)$.

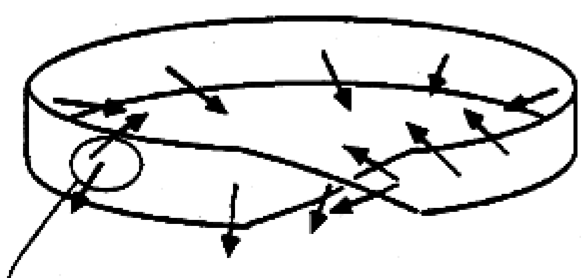


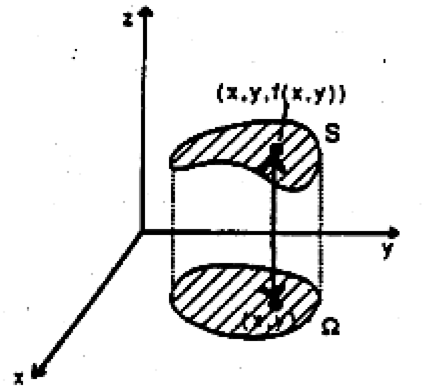
Note that we always have a choice of unit normal $\vec{n}$ (out of two possibilities) and the sign of our answer is determined by our choice of normal. Given a surface S we must make this choice of normal consistently. We call this choice of normal the *orientation* of the surface.[1] Given an oriented surface S, the flux of F across S, as defined above, is uniquely determined. (The orientation of S tells us which $\vec{n}$ to use.)

One problem in calculating the flux of $\overrightarrow{\mathbf{F}}$ through S, denoted $\iint_S \overrightarrow{\mathbf{F}} \cdot \vec{n}\, d\sigma$, is determining $d\sigma$, the surface area element. We'll do this in the case that our surface S is given as the graph of a continuous function, $z = f(x, y)$.

Let Ω be the shadow of S in the xy-plane. There is a one-to-one correspondence between points (x, y) in Ω and points $(x, y, f(x, y))$ on S.

[1] Not every surface admits an orientation. For instance, a Möbius strip has only one "side:" if you choose a normal vector at one point and move along the strip, always making a consistent choice of normal vector, when you get back to where you started, you end up with the opposite vector you started with, a clear inconsistency! One cannot define an orientation on a Möbius strip. However, most surfaces you will encounter will be orientable.





Partition 'most'[2] of the region Ω lying in the xy-plane into tiny rectangles of area $\Delta A = \Delta x \Delta y$. Each tiny rectangle in the plane corresponds to a small piece of S whose area will be denoted by $\Delta\sigma$. We want to find out what $\Delta\sigma$ is in terms of ΔA. (The area of $\Delta\sigma$ is always greater than or equal to that of ΔA. Why? When would equality hold?)

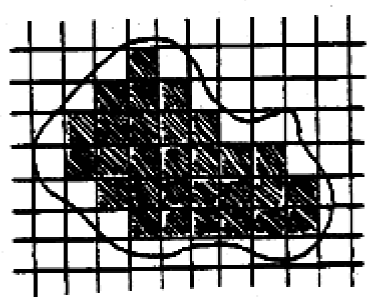
If the rectangle in Ω is really tiny, the corresponding piece of S will be almost flat. We can approximate it by a piece of the tangent plane to S.

Let's begin by considering the following parallelogram. Three of its vertices are

$$P = (x, y, f(x,y)), \quad Q = (x + \Delta x, y, f(x + \Delta x, y)), \quad \text{and} \quad R = (x, y + \Delta y, f(x, y + \Delta y)),$$

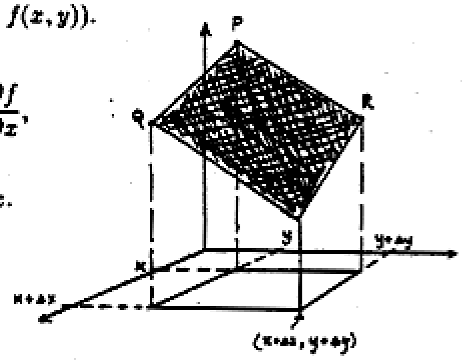
the points on S corresponding to the points (x,y), $(x + \Delta x, y)$, and $(x, y + \Delta y)$ in Ω.

[2] As you make your partition finer and finer, more and more of the region will be included in the partition.



The parallelogram is spanned by the vectors $\overrightarrow{\mathbf{PQ}}$ and $\overrightarrow{\mathbf{PR}}$ where

$$\overrightarrow{\mathbf{PQ}} = \mathbf{Q} - \mathbf{P} = (\Delta x, 0, f(x+\Delta x, y) - f(x,y)).$$



Now, recall that

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \frac{\partial f}{\partial x},$$

so for small Δx

$$f(x+\Delta y, y) - f(x,y) \approx \frac{\partial f}{\partial x}\Delta x.$$

Therefore

$$\overrightarrow{\mathbf{PQ}} \approx \left(\Delta x, 0, \frac{\partial f}{\partial x}\Delta x\right).$$

Similarly,

$$\overrightarrow{\mathbf{PR}} = \mathbf{R} - \mathbf{P} \approx \left(0, \Delta y, \frac{\partial f}{\partial y}\Delta y\right).$$

The area $\Delta\sigma$ is approximately equal to the area of the parallelogram spanned by $\overrightarrow{\mathbf{PR}}$ and $\overrightarrow{\mathbf{PQ}}$.

$$\Delta\sigma \approx |\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}}| \approx \left|\left(\Delta x, 0, \frac{\partial f}{\partial x}\Delta x\right) \times \left(0, \Delta y, \frac{\partial f}{\partial y}\Delta y\right)\right| = \left|\left(-\frac{\partial f}{\partial x}\Delta x\Delta y, -\frac{\partial f}{\partial y}\Delta x\Delta y, \Delta x\Delta y\right)\right|$$

i.e.,

$$\Delta\sigma \approx |(-f_x, -f_y, 1)\,\Delta x\Delta y| = |(-f_x, -f_y, 1)|\,\Delta x\Delta y$$
$$\Delta\sigma \approx \sqrt{(f_x)^2 + (f_y)^2 + 1}\,\Delta A.$$

[Note that we have approximated the area of a parallelogram which is tangent to our surface at the point P.] Our conjecture (which can be verified by a more rigorous argument) is:

$$d\sigma = |(-f_x, -f_y, 1)|\,dA = |(-f_x, -f_y, 1)|\,dx\,dy.$$

Also, since $\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} = (-f_x, -f_y, 1)\Delta x\,\Delta y$, we know that $(-f_x, -f_y, 1)$ must be *normal* to the parallelogram spanned by $\overrightarrow{\mathbf{PQ}}$ and $\overrightarrow{\mathbf{PR}}$ (though it is not a unit normal unless $f_x = f_y = 0$).[3] We can make the *unit* normal $\vec{n}$ as follows:

$$\vec{\mathbf{n}} = \pm\frac{(-f_x, -f_y, 1)}{|(-f_x, -f_y, 1)|}.$$

Then, using the expressions for $\vec{\mathbf{n}}$ and $d\sigma$ found above, we see that:

$$\text{The Flux of } \vec{\mathbf{F}} \text{ through } S = \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\,d\sigma = \pm\iint_\Omega \vec{\mathbf{F}} \cdot \frac{(-f_x, -f_y, 1)}{|(-f_x, -f_y, 1)|}|(-f_x, -f_y, 1)|dA$$

[3]Notice that if we let $g(x,y,z) = z - f(x,y)$, then $|(-f_x, -f_y, 1)|dx\,dy = |\vec{\nabla}g|dx\,dy$. So $d\sigma = |\vec{\nabla}g|dA$ where $d\sigma$ corresponds to the area $\Delta\sigma$ on the surface S, and dA corresponds to the ΔA area of the corresponding projection onto the xy- plane. We see that the magnitude of $\vec{\nabla}g$ gives us the distortion factor for the surface area element. (Notice that $|\vec{\nabla}g|$ is always greater than or equal to one.)

$$= \pm \iint_{\Omega} \overrightarrow{\mathbf{F}} \cdot (-f_x, -f_y, 1)\, dx\, dy$$

for a surface S given by $z = f(x, y)$, with projection Ω in the xy-plane.

CAUTION: Don't get confused between $(-f_x, -f_y, 1)$ and the unit normal $\vec{\mathbf{n}}$. Typically $(-f_x, -f_y, 1)$ is not a unit vector, although it is is parallel to $\vec{\mathbf{n}}$.

Note: Every surface of the form $z = f(x, y)$ has an unambiguous top and bottom side. Its orientation can be with either upward pointing normals (in the direction of $(-f_x, -f_y, 1)$) or downward pointing normals (in the direction of $(f_x, f_y, -1)$).

For example, if S is given by $z = f(x, y)$, oriented with upward normals, we have

$$\iint_S \overrightarrow{\mathbf{F}} \cdot \vec{\mathbf{n}} = \iint_{\Omega} \overrightarrow{\mathbf{F}} \cdot (-f_x, -f_y, 1)\, dA$$

(where Ω is the projection of S in the xy-plane).

Important! Every occurrence of z in the right hand side must be replaced with $f(x, y)$ before integrating.

We get similar formulas for surfaces of the form $x = f(y, z)$ or $y = f(x, z)$. For example, if S is given by $y = f(x, z)$ oriented with normals pointing to the $-y$ side, we use $(f_x, -1, f_z)$ for our (non-unit) normal vector, and

$$\iint_S \overrightarrow{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma = \iint_{\Omega} \overrightarrow{\mathbf{F}} \cdot (f_x, -1, f_z)\, dA$$

(where Ω is the shadow in the xz-plane and $dA = dx\, dz$.) And so on for the other possibilities. [The sign of the "1" indicates the choice of orientation.]

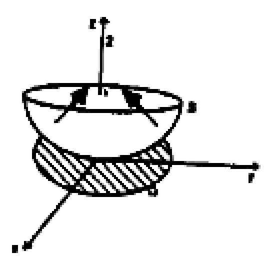
Of course, many surfaces can't be represented as the graph of a single function of two variables. No closed surface, like the unit sphere, for example, can be. But we can break the sphere into two hemispheres: $z = +\sqrt{1 - x^2 - y^2}$ and $z = -\sqrt{1 - x^2 - y^2}$, and work on each piece separately (making sure that our orientation of the whole surface is compatible with our choice of normals on each piece). We can use similar tactics with other surfaces. For a more general discussion of parametrized surfaces, see *Vector Calculus* by Jerrold Marsden and Anthony Thomba.

Applications and Examples

We now turn to some examples of calculating flux.

Example 1: Find the flux of the vector field $\mathbf{F} = (x + z, y, y - z)$ across the part of the paraboloid $z = x^2 + y^2$ with $z \leq 1$, oriented with inward normals.

Solution:



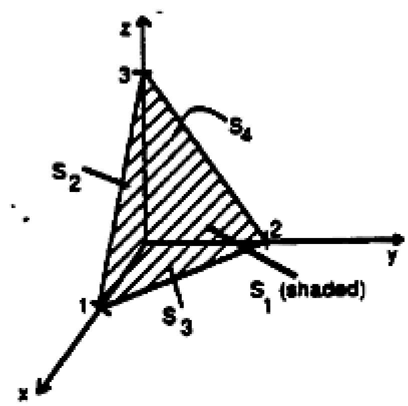
S is already given in the form $z = f(x, y)$. The region Ω is the unit disk in the xy-plane. (It has radius 1 because the surface has boundary where $z = 1$, i.e., where $x^2 + y^2 = 1$.) The inward-pointing normals have positive k- components, so we use the vector $(-f_x, -f_y, +1) = (-2x, -2y, 1)$.

$$\iint_S \overrightarrow{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma = \iint_\Omega (x+z, y, y-z)\cdot(-2x, -2y, 1)dA = \iint_\Omega (x+x^2+y^2)(-2x)+y(-2y)+(y-x^2-y^2)dA$$

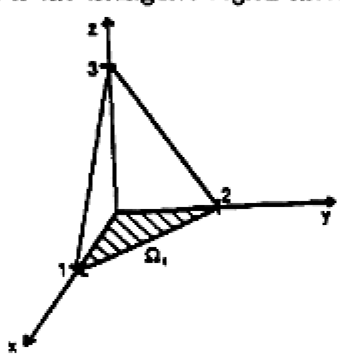
where we have replaced z by $x^2 + y^2$ in the final integral. The calculation of the integral is left as an exercise. (Use polar coordinates. Symmetry arguments can simplify the computation a little, but not dramatically.)

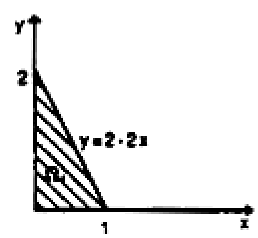
Example 2: Find the inward flux of $\overrightarrow{\mathbf{F}} = (x, z, y)$ across the boundary surface of the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

Solution: The surface has four faces; we must find the flux through each separately. (Later we'll encounter the Divergence Theorem which will make this problem easier, just as Green's Theorem made integrals around closed curves easier.) Let's start with the slanted face, which we'll call S_1.



It is part of the plane through $(1,0,0)$, $(0,2,0)$, $(0,0,3)$, i.e., the plane $6x + 3y + 2z = 6$. (You can get this equation from the general procedure for finding an equation of the plane through three given points, or by inspection.) We can therefore write the slant face as $z = 3 - 3x - \frac{3}{2}y$. The shadow Ω_1 in the xy-plane is the triangular region sketched below.





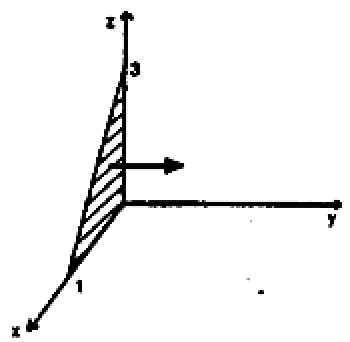
The inward-pointing normal points downward for the slanted face, so we use $(+f_x, +f_y, -1)$, which is $(-3, -\frac{3}{2}, -1)$. We get

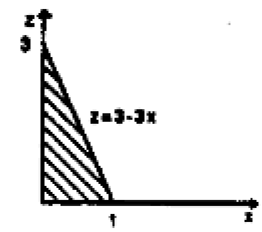
$$\iint_{S_1} \vec{\mathbf{F}} \cdot \bar{\mathbf{n}}\, d\sigma = \iint_{\Omega_1} (x,z,y) \cdot \left(-3, -\frac{3}{2}, -1\right) dA = \iint_{\Omega_1} \left[-3x - \left(\frac{3}{2}\right)\left(3 - 3x - \frac{3y}{2}\right) - y\right] dA$$

(substituting $3 - 3x - \frac{3y}{2}$ for z)

$$= \int_0^1 \int_0^{2-2x} \left(-\frac{9}{2} + \frac{3x}{2} + \frac{5y}{4}\right) dy\, dx = \ldots.$$

Now consider the face S_2 in the xy-plane. Its equation is just $y = f(x,z) = 0$ since it's part of the $y = 0$ plane. (Don't let the triangular boundary confuse you into thinking you need a more complicated equation!) The shadow Ω_2 is just S_2 itself, since S_2 is already in the xz-plane. We want normals pointing to the $+y$ side, i.e., $(-f_x, +1, -f_z) = (0,1,0)$. So,



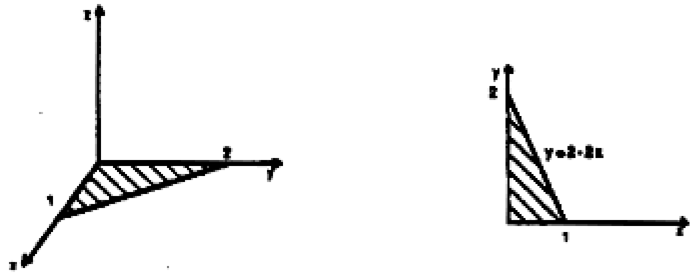


$$\iint_{S_2} \vec{\mathbf{F}} \cdot \bar{\mathbf{n}} d\sigma = \iint_{\Omega_2} (x,z,y) \cdot (0,1,0) dA = \int_0^2 \int_0^{3-3x} z\, dz\, dx = \ldots$$

Note that here $(-f_x, 1, -f_z)$ actually is the unit normal. This is because S_2 is already in the xz-plane, so $d\sigma$ and dA are the same ($d\sigma$ means a small piece of area on the surface and dA means a small piece of area on the shadow). In this simple case, the flux integral is the same as the double integral of $\vec{\mathbf{F}} \cdot \bar{\mathbf{n}}$.

Similarly, consider the face S_3 in the xy-plane. It's already in the xy-plane, so $d\sigma = dA$, and $\bar{\mathbf{n}} = +\vec{\mathbf{k}}$ since the upward normal is the inward-pointing one. The flux is therefore

$$\iint_{S_3} \vec{\mathbf{F}} \cdot \bar{\mathbf{n}}\, d\sigma = \iint_{S_3} (x,z,y) \cdot (0,0,1) dA = \int_0^1 \int_0^{2-2x} y\, dy\, dx = \ldots$$



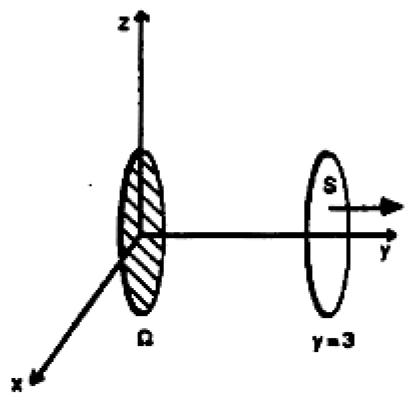
For S_4, the face in the yz-plane, $\vec{n} = +\vec{i}$ and we have

$$\iint_{S_4} \vec{\mathbf{F}} \cdot \vec{n}\, d\sigma = \iint_{\Omega_4} (x, z, y) \cdot (1,0,0)\, dA = \iint_{\Omega_4} x\, dy\, dz = \iint_{\Omega_4} 0\, dy\, dz = 0$$

since $x = 0$ on the yz-plane. (Remember, even if x were nonzero, we'd still have to write it as the appropriate function of y and z before doing the integral.)

Example 3: Find the flux of $\vec{\mathbf{F}} = (x, y - x, x + z)$ through the unit disk in the $y = 3$ plane (centered at the y-axis), oriented with normals in the $+y$ direction.

Solution:



The equation of the surface is $y = 3$, since it's part of the $y = 3$ plane. [Don't let the fact that its boundary is circular confuse you into writing $x^2 + z^2 = 1$ or something along those lines. The boundary will be described by the bounds of integration.] So $y = f(x,z) = 3$, and $f_x = f_z = 0$. The shadow Ω is the unit disk in the xz-plane. We use the vector $(-f_x, +1, -f_z) = (0,1,0)$ since the normals are supposed to be pointing in the $+y$ direction.

$$\iint_{S} \vec{\mathbf{F}} \cdot \vec{n}\, d\sigma = \iint_{\Omega} (x, y - x, x + z) \cdot (0,1,0)\, dA = \iint_{\Omega} (3 - x)\, dA$$

(we've plugged in 3 for y, remember). We can use symmetry arguments to assert that

$$\iint_{\Omega} -x\,dA = 0$$

(For each positive value of x in the unit disk, there's a corresponding negative value of x on the other side of the disk which will cancel it out.) So we're left with

$$\iint_{\Omega} 3dA = 3\ (\text{area of } \Omega)\ = 3\pi.$$

Sometimes one can find a flux by geometric reasoning without doing the integral. For example, if $\vec{\mathbf{F}} \cdot \vec{n}$ is zero everywhere, we know the flux is zero. If $\vec{\mathbf{F}} \cdot \vec{n}$ is a constant, say c, and if we know the surface area of S geometrically, then the flux is c times the surface area. This makes sense given our original development of flux, splitting the surface into small pieces, taking $\vec{\mathbf{F}} \cdot \vec{n}$ $(= c)$ times the area of each small piece, and adding them up. This gives c times the sum of the areas of the pieces, i.e., c times the total area.

Example 4: Find the outward flux of $\vec{\mathbf{F}} = (3x, 3y, 3z)$ through the sphere of radius 2 centered at the origin.

Solution: We can do this geometrically. To find the unit normal $\vec{n}$, there's no need to break the sphere into hemispheres and write $z = \pm\sqrt{4 - x^2 - y^2}$. We know the outward normal direction is radial (why?), and thus is given by (x, y, z). The length of (x, y, z) is $\sqrt{x^2 + y^2 + z^2} = 2$ since $x^2 + y^2 + z^2 = 4$ on the sphere. Dividing by the length, we get the outward unit normal vector $\vec{n} = \frac{(x,y,z)}{2}$.

Next we calculate $\vec{\mathbf{F}} \cdot \vec{n}$. We get $(3x, 3y, 3z) \cdot (\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) = \frac{3}{2}(x^2 + y^2 + z^2) = \frac{3}{2} \cdot 4 = 6$ since $x^2 + y^2 + z^2 = 4$ on the sphere. 6 is indeed a constant! So the flux is 6 times the surface area of the sphere, i.e., $(6)(16\pi) = 96\pi$.

IMPORTANT CAUTIONARY NOTE: If you find $\vec{n}$ and calculate $\vec{\mathbf{F}} \cdot \vec{n}$ and find that it's not constant, so that you have to use the above formulas to calculate the flux integral directly, there may be a great temptation to integrate what you just found for $\vec{\mathbf{F}} \cdot \vec{n}$ over Ω. *This is wrong!* You must integrate $\vec{\mathbf{F}} \cdot (-f_x, -f_y, 1)$ (or whichever one is appropriate) over Ω, not $\vec{\mathbf{F}} \cdot \vec{n}$. $\vec{\mathbf{F}} \cdot \vec{n}$ is the quantity that needs to be integrated over the *surface*. When you do the double integral over the projection, you need an extra factor to take into account the fact that areas on S are in general bigger than areas on the projection. The vectors $(-f_x, -f_y, 1)$ take this into account. (They are always as long or longer than unit normals.) Equivalently, you can multiply $\vec{\mathbf{F}} \cdot \vec{n}$ by $d\sigma$ and integrate over Ω.

In other words, you use the *unit* normal only when using geometric reasoning involving the surface area of the surface to find the *surface integral*; when you are calculating a flux by doing a double integral over Ω, *the shadow region* (or domain of the parametrization), you use the (not-necessarily-unit) normal $(-f_x, -f_y, 1)$ (or its appropriate analogue depending on which coordinates are your parameters and what the orientation is).

For example, in the above problem, if you didn't realize that $\overrightarrow{F} \cdot \vec{n} = \frac{3}{2}(x^2 + y^2 + z^2)$ was constant and went to do the calculation directly,

$$\iint_S \frac{3}{2}\left(x^2 + y^2 + z^2\right) d\sigma.$$

You could write the sphere on the two hemispheres $z = +\sqrt{4 - x^2 - y^2}$ with upward normals and $z = -\sqrt{4 - x^2 - y^2}$ with downward normals. The flux through the top hemisphere is

$$\iint_\Omega (3x, 3y, 3z) \cdot (-f_x, -f_y, 1)\, dA = \iint_\Omega \left[3x^2(4 - x^2 + y^2)^{-\frac{1}{2}} + 3y^2(r - x^2 + y^2)^{-\frac{1}{2}} + 3z\right] dA$$

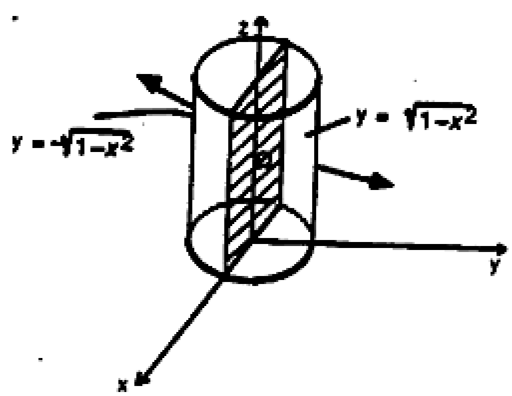
plug in $z = \sqrt{4 - x^2 - y^2}$, and evaluate. The flux through the bottom is

$$\iint_\Omega (3x, 3y, 3z) \cdot (f_x, f_y, -1)\, dA = \iint_\Omega \left[3x^2(4 - x^2 + y^2)^{-\frac{1}{2}} + 3y^2(4 - x^2 + y^2)^{-\frac{1}{2}} - 3z\right] dA$$

plug in $z = -\sqrt{4 - x^2 - y^2}$, and evaluate. Then you add up the fluxes for the two hemispheres, and you should get 96π. (If you're wondering why the integands on the right hand side look almost the same when in one case we have $(-f_x, -f_y, 1)$ and in the other $(f_x, f_y, -1)$, it's because the f's differ by a sign.)

Example 5: Find the flux of $\mathbf{F} = (z, y, e^x)$ out of the part of the cylinder $x^2 + y^2 = 1$ with $0 \leq z \leq 2$.

Solution: We can break the cylinder into two halves: $y = +\sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$. In each case, Ω is the rectangle $-1 \leq x \leq 1$, $0 \leq z \leq 2$ in the xz-plane.



For the right half of the cylinder, we want the $+y$ normals, so we evaluate

$$\iint_\Omega (z, y, e^x) \cdot (-f_x, 1, -f_z)\, dz\, dx = \iint_\Omega \left(z, \sqrt{1 - x^2}, e^x\right) \cdot \left(x\left(1 - x^2\right)^{-\frac{1}{2}}, 1, 0\right) dz\, dx$$

$$= \int_{-1}^{1}\int_{0}^{2}\left[xz\left(1-x^2\right)^{-\frac{1}{2}} + \sqrt{1-x^2}\right] dz\,dx$$

$$= \int_{-1}^{1}\int_{0}^{2} xz(1-x^2)^{-\frac{1}{2}} dz\,dx + \int_{-1}^{1}\int_{0}^{2}\sqrt{1-x^2}\,dz\,dx.$$

By symmetry arguments the left-hand integral is zero. (For each fixed z, the integrand is an odd function of x, i.e., $\frac{(-x)z}{\sqrt{1-(-x)^2}} = -\frac{xz}{\sqrt{1-x^2}}$ over the region of integration; for every positive x there is a corresponding negative x of equal magnitude, so the total integral is zero.) Thus, we're left with

$$\int_{-1}^{1}\int_{0}^{2}\sqrt{1-x^2}\;dz\,dx = \int_{-1}^{1} 2\sqrt{1-x^2}\;dx$$

$$= 2(\text{one-half the area of a unit circle }) = 2\cdot\frac{1}{2}\pi = \pi.$$

For the left half of the cylinder, $f(x,z) = -\sqrt{1-x^2}$ and we use $-y$ normals, so we get

$$\iint_{\Omega}(z,y,e^x)\cdot(-f_x,1,-f_z)\,dz\,dx = \iint_{\Omega}\left(z,\sqrt{1-x^2},e^x\right)\cdot\left(x\left(1-x^2\right)^{-\frac{1}{2}},1,0\right)dz\,dx,$$

which boils down to the same integral as before (that's only because the vector field happens to be somewhat symmetrical; this doesn't always happen). Thus the total flux is $\pi + \pi = 2\pi$.