# 1 Midterm 1 Answer Sheet

## 1.1 Part I

1) **Answer is (c)**

$$A \cdot B = (-2\mathbf{i} + (t-1)\,\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{j} + t\mathbf{k}) = (t-1) + 2t = 3t - 1 = 0 \Rightarrow$$
$$\Rightarrow t = \frac{1}{3}$$

2) **Answer is (d)**

$$A \times B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & t-1 & 2 \\ 0 & 1 & t \end{vmatrix} = \left(t^2 - t - 2\right)\mathbf{i} + 2t\mathbf{j} - 2\mathbf{k}$$

which is nonzero for any value of $t$.

3) **Answer is (b)**

$$(A+B) \cdot (A \times B) = A \cdot (A \times B) + B \cdot (A \times B) = 0 + 0 = 0$$

since both $A$ and $B$ are normal to $(A \times B)$.

4) **Answer is (e)**

Consider the level curve passing through $(0,0)$. The first three choices are eliminated because this level curve would not have points in the first quadrant (where both $x$ and $y$ are positive). By checking some points in the graph and using the symmetries, (d) and (f) can be eliminated.

5) **Answer is (b)**

The fact that the curve is an ellipse eliminates (a), (c) and (e). Since (d) would have longer axis in the $y$ direction, we can eliminate it as well.

6) **Answer is (a)**

$$\mathbf{r}(t) = e^{\sqrt{2}t}\cos t\,\mathbf{i} + e^{\sqrt{2}t}\sin t\,\mathbf{j}$$
$$\mathbf{r}'(t) = \left(\sqrt{2}e^{\sqrt{2}t}\cos t - e^{\sqrt{2}t}\sin t\right)\mathbf{i} + \left(\sqrt{2}e^{\sqrt{2}t}\sin t + e^{\sqrt{2}t}\cos t\right)\mathbf{j}$$
$$|\mathbf{r}'(t)| = \left(\left(\sqrt{2}e^{\sqrt{2}t}\cos t - e^{\sqrt{2}t}\sin t\right)^2 + \left(\sqrt{2}e^{\sqrt{2}t}\sin t + e^{\sqrt{2}t}\cos t\right)^2\right)^{\frac{1}{2}} =$$
$$= \left(2e^{2\sqrt{2}t} + e^{2\sqrt{2}t}\right)^{\frac{1}{2}} = \sqrt{3}e^{\sqrt{2}t}$$

so the arc length is

$$L = \int_0^\pi |\mathbf{r}'(t)|\, dt = \int_0^\pi \sqrt{3} e^{\sqrt{2}t} dt = \sqrt{\frac{3}{2}} e^{\sqrt{2}t}\Big|_0^\pi = \sqrt{\frac{3}{2}} \left(e^{\sqrt{2}\pi} - 1\right)$$

7) **Answer is (e)**
Using the differential of $f$ and the partial derivatives at $(1, 2, 3)$:

$$\frac{df}{f} = 2\frac{dx}{x} + 3\frac{dy}{y} - \frac{dz}{z}$$

So

$$\left|\frac{\Delta f}{f}\right| \le 2\left|\frac{\Delta x}{x}\right| + 3\left|\frac{\Delta y}{y}\right| + \left|\frac{\Delta z}{z}\right| \le 0.002 + 0.003 + 0.001 = 0.006$$

that corresponds to 0.6%. Note that we must use the absolute values to get the maximum error in $f$ (since we do not know if $\frac{\Delta z}{z} = \pm 0.1\%$, or anything in between.

8) **Answer is (e)**
It is easy to see that $z(0, 0) = 1$. We can implicitly differentiate

$$ze^{\left(x^2-y^2\right)/z} = 1$$

with relation to $x$ to get

$$\frac{\partial z}{\partial x} e^{\left(x^2-y^2\right)/z} + ze^{\left(x^2-y^2\right)/z} \left(\frac{2x}{z} - \frac{x^2 - y^2}{z^2}\frac{\partial z}{\partial x}\right) = 0$$

which gives $\frac{\partial z}{\partial x} = 0$ when $x = y = 0$.

**Another solution:**
We can try to solve for $z$ (note that $z$ must be positive, so all operations are valid)

$$e^{\left(x^2-y^2\right)/z} = \frac{1}{z} \Rightarrow \frac{x^2 - y^2}{z} = \ln\left(\frac{1}{z}\right) = -\ln z \Rightarrow$$
$$\Rightarrow x^2 - y^2 = -z \ln z$$

We still can not solve for $z$, but the implicit differentiation is now faster:

$$2x = \left(-\ln z - z\frac{1}{z}\right)\frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = -\frac{2x}{1 + \ln z}$$
$$-2y = \left(-\ln z - z\frac{1}{z}\right)\frac{\partial z}{\partial y} \Rightarrow \frac{\partial z}{\partial y} = \frac{2y}{1 + \ln z}$$

so clearly

$$\frac{\partial z}{\partial x}(0, 0) = \frac{\partial z}{\partial y}(0, 0) = 0$$

## 1.2 Part II

### Question 1

a) The plan $\Pi$ passes through $A = (0, -2, 1)$, $B = (2, 0, 1)$ and $C = (0, 0, 2)$, therefore it contains the vectors $\mathbf{v} = \overrightarrow{AB} = 2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{w} = \overrightarrow{AC} = 2\mathbf{j} + \mathbf{k}$. Then the normal vector to $\Pi$ is

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{j}$$

therefore the equation of $\Pi$ is (using $(0, 0, 2)$)

$$2(x-0) - 2(y-0) + 4(z-2) = 0 \Rightarrow$$
$$\Rightarrow x - y + 2z = 4$$

b) The line passing through $(0, -1, 3)$ and $(3, 5, 0)$ has direction given by the vector

$$(3\mathbf{i} + 5\mathbf{j}) - (-\mathbf{j} + 3\mathbf{k}) = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

Using the point $(3, 5, 0)$, the parametrized equations of the line with parameter $t$ are

$$x(t) = 3 + 3t$$
$$y(t) = 5 + 6t$$
$$z(t) = -3t$$

c) To find the intersection of the line with the plane $\Pi$, substitute the equations of the line into the equation of $\Pi$:

$$(3 + 3t) - (5 + 6t) - 6t = 4 \Rightarrow 3t = -2$$

therefore the point is $(1, 1, 2)$.

d) To find the acute angle between the line and the normal vector to $\Pi$, we use

$\mathbf{A} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ as a normal vector to the plane (from (a))
$\mathbf{B} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ as a vector in the direction of the line (from (b))

and $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta$ to find

$$\cos\theta = \frac{(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{|\mathbf{i} - \mathbf{j} + 2\mathbf{k}|\,|\mathbf{i} + 2\mathbf{j} - \mathbf{k}|} = \frac{1 - 2 - 2}{\sqrt{6}\sqrt{6}} = -\frac{1}{2}$$

so the obtuse angle is $\frac{2\pi}{3}$, and the acute angle is $\frac{\pi}{3}$.

## Question 2

a) The path is $x = \cos t$ and $y = \sin t$, $0 \le t \le 2\pi$. Using the chain rule

$$\begin{aligned}\frac{dh}{dt} &= \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = \\ &= (-7\cos t + 2\sin t)(-\sin t) + (2\cos t - 3\sin t)(\cos t) = \\ &= 2\cos^2 t - 2\sin^2 t + 4\sin t\cos t = 2\cos 2t + 2\sin 2t\end{aligned}$$

b) The points with $\frac{dh}{dt} = 0$ correspond to

$$\cos 2t + \sin 2t = 0 \Rightarrow \tan 2t = -1$$

Since $0 \le 2t \le 4\pi$, we have

$$2t = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \text{ so } t = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$$

c) The contour line that the second hiker follows is given by

$$h(x,y) = h(1,-0.5) = \text{constant}$$

Differentiate to get the slope of the tangent line to the contour curve at $(1,-0.5)$

$$dh = 0 = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy \Rightarrow \frac{dy}{dx} = -\frac{\partial h/\partial x}{\partial h/\partial y}$$

But since

$$h_x(1,-0.5) = -8 \text{ and } h_y(1,-0.5) = \frac{7}{2}$$

then

$$\frac{dy}{dx} = \frac{16}{7}$$

gives the direction that the hiker follows.

*Note 1: It is not true that the second hiker follows the same circle that the first one followed. There is no reason to believe that the level curves of $h$ are circles.*

*Note 2: The position vector $r$ of the hiker and the function $h$ are very different things; there is no reason to believe that the hiker follows the direction of $\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\mathbf{i} + \frac{\partial \mathbf{h}}{\partial \mathbf{y}}\mathbf{j}$. Actually, now that you know that this is the gradient of $h$, it should be clear that the direction along a level curve is actually orthogonal to this vector.*

## Question 3

a)

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{r} \times \left(-\frac{k}{m}\mathbf{r}\right) = \mathbf{0}$$

since

$$\mathbf{r} \times \mathbf{r} = \mathbf{r}' \times \mathbf{r}' = \mathbf{0}$$

Hence $\mathbf{r}(t) \times \mathbf{r}'(t)$ is constant, and the ball's motion lies in the plane through the origin perpendicular to $\mathbf{r}(t) \times \mathbf{r}'(t)$ (or on a fixed line through the origin, if $\mathbf{r}(t) \times \mathbf{r}'(t) = 0$).

*Notes: Alternatively, one could conclude that the position vector sweeps equal areas at equal amounts of time.*

*It does **not** follow that the ball moves in an elliptic orbit. In our particular case, it is true that the orbit is elliptic, but that has nothing to do with the constancy of $\mathbf{r}(t) \times \mathbf{r}'(t)$. The orbits are generally not circular (although they can be on occasion, as in part c).*

*Two common errors were the following incorrect rules for differentiating a cross product:*

$$\frac{d}{dt}(\mathbf{A}(t) \times \mathbf{B}(t)) = \mathbf{A}'(t) \times \mathbf{B}'(t)$$

*and*

$$\frac{d}{dt}(\mathbf{A}(t) \times \mathbf{B}(t)) = \mathbf{A}'(t) \times \mathbf{B}(t) + \mathbf{B}'(t) \times \mathbf{A}(t).$$

*The first of these is a more serious mistake than the second.*

b)

$$
\begin{aligned}
E(t) &= \frac{m}{2}(\mathbf{r}'\cdot\mathbf{r}') + \frac{k}{2}(\mathbf{r}\cdot\mathbf{r}) \\
E'(t) &= m(\mathbf{r}''\cdot\mathbf{r}') + k(\mathbf{r}'\cdot\mathbf{r}) = \\
&= m\left(-\frac{k}{m}\mathbf{r}\cdot\mathbf{r}'\right) + k(\mathbf{r}'\cdot\mathbf{r}) = 0
\end{aligned}
$$

since again

$$\mathbf{r}'' = -\frac{k}{m}\mathbf{r}$$

This shows that $E(t)$ is a constant independent of $t$.

*Notes: One common mistake was the attractive but incorrect equation*

$$\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$$

*(If this equation looks believable, think about circular motion.) Another common mistake was assuming that* $\mathbf{r}(t)$ *and* $\mathbf{r}''(t)$ *are perpendicular to* $\mathbf{r}'(t)$*, as they would be in uniform circular motion. A similar error was directly assuming that* $|\mathbf{r}(t)|$ *and* $|\mathbf{r}'(t)|$ *are constant.*

*In some solutions, equations ended up being written in which vectors and scalars were combined in illegitimate ways, e.g.,* $\mathbf{r}(t)\cdot\mathbf{r}(t)\cdot\mathbf{r}(t)$ *or* $m\mathbf{r}''(t)+k|\mathbf{r}(t)|$*, which make no sense. It's a good idea to keep an eye out for this problem, since it is a sure sign that something has gone wrong.*

*After grading this problem, I've got three comments about writing style. First, clearly indicate all dot and cross products. (Some people simply juxtaposed vectors with no hint as to how they were being combined.) Second, don't keep pointless absolute value signs. (Some people wrote* $|\mathbf{r}(t)|^2 = |\mathbf{r}(t)\cdot\mathbf{r}(t)|$*. That's true, but you need to get rid of the unnecessary absolute value signs on the right sooner or later.) Third, keep the logic of the solution clear.*

*The third point is worth expanding on. Some people began by assuming* $E(t)$ *was constant, then manipulated equations, and finally concluded with a true statement; they deduced from this that* $E(t)$ *really was constant. That's nonsense, logically. (It doesn't matter that you can deduce true statements from the constancy of* $E(t)$*; what matters is whether you can deduce the constancy of* $E(t)$ *from true statements.) This argument is simple enough that doing it backwards is unlikely to lead to confusion, but it is a bad habit that can cause serious problems in more complicated cases.*

c) From (a), we know that

$$\mathbf{r}\left(T\right)\times\mathbf{r}'\left(T\right)=\mathbf{r}\left(0\right)\times\mathbf{r}'\left(0\right)=\mathbf{i}\times\mathbf{j}=\mathbf{k}$$

so

$$-\mathbf{j}\times\mathbf{r}'(T)=\mathbf{k}$$

**This is not enough to state that $\mathbf{r}'(T)=\mathbf{i}$.** Indeed, if we write $\mathbf{r}'(T)=A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$, then

$$-\mathbf{j}\times\mathbf{r}'(T)=-\mathbf{j}\times\left(A\mathbf{i}+B\mathbf{j}+C\mathbf{k}\right)=A\mathbf{k}-C\mathbf{i}$$

so we have $A=1$ and $C=0$, but we can't determine $B$ just yet, that is

$$\mathbf{r}'(T)=\mathbf{i}+B\mathbf{j}$$

Now, from (b)

$$\begin{aligned}&E(0)=E(T)\Rightarrow\\&\Rightarrow\frac{m}{2}.1+\frac{k}{2}.1=\frac{m}{2}.\left|\mathbf{r}'(T)\right|^2+\frac{k}{2}.1\Rightarrow\\&\Rightarrow\left|\mathbf{r}'(T)\right|=1\end{aligned}$$

and this allows us to pin down the value of $B$ as 0, that is

$$\mathbf{r}'(T)=\mathbf{i}$$

*Notes: Some solutions used the circularity of the motion. In this particular case, it turns out that the motion is circular. However, we do not know that a priori, so any solution that uses that fact must explain why it holds here.*

*It does not follow from $\mathbf{r}(0)=\mathbf{i}$ and $\mathbf{r}'(0)=\mathbf{j}$ that the motion is circular. Under those conditions, the motion is given by*

$$\mathbf{r}(t)=\cos\left(\sqrt{\frac{k}{m}}t\right)\mathbf{i}+\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)\mathbf{j}$$

*and is elliptical. Adding the constraint that $\mathbf{r}(T)=-\mathbf{j}$ for some $T$ forces $m=k$ and gives circular motion.*