Math 21a, Fall 1998
Second Midterm – Solutions

1 Part I

1) Answer: a)
The hyperboloid is the 0-level surface of $f(x,y,z) = x^2 + y^2 - z^2 + 3$. Thus Π is described by the linear equation

$$\begin{aligned} 0 &= (x-2)\frac{\partial f}{\partial x}(2,3,4) + (y-3)\frac{\partial f}{\partial y}(2,3,4) + (z-4)\frac{\partial f}{\partial z}(2,3,4) \\ &= 4(x-2) + 6(y-3) - 8(z-4) \;=\; 4x + 6y - 8z + 6 \end{aligned}$$

or equivalently, by

$$h(x,y,z) \;=\; -3\,, \qquad \text{where} \qquad h(x,y,z) = 2x + 3y - 4z\,.$$

But $h(1,1,1) = 1$ and $h(1,1,-1) = 9$ are both strictly greater than -3. Conclusion: neither P nor Q lie on Π, and they are on the same side.

2) Answer: a)
The function $f(x,y)$ has a critical point at (x,y) precisely when

$$0 \;=\; \frac{\partial f}{\partial x} \;=\; 6x - 4y \qquad \text{and} \qquad 0 \;=\; \frac{\partial f}{\partial y} \;=\; -4x + 2y\,,$$

which happens only at the origin. The second derivative test at $x = y = 0$ gives

$$\left.\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}\right)\right|_{x=y=0} \;=\; 6 \cdot 2 - 4^2 \;<\; 0 \quad \Longrightarrow \quad \text{saddle point}\,,$$

so f has no local maxima or local minima in the interior of the disk. On the other hand, this disk is closed and bounded, forcing the existence of at least one global minimum and at least one global maximum on the disk.

3) Answer: f)
To find the critical point(s), we solve the simultaneous equations

$$0 \;=\; \frac{\partial f}{\partial x} \;=\; -6x^2 + 6xy \qquad \text{and} \qquad 0 \;=\; \frac{\partial f}{\partial y} \;=\; 3x^2 + 9y^2 - 3\,.$$

By assumption, $x > 0$, permitting us to divide the first equation by x, hence $x = y$. Plugging this into the second equation gives $12x^2 = 3$, hence $x = y = \pm\frac{1}{2}$. The restriction $x > 0$ only leaves the single point $x = y = \frac{1}{2}$ to be considered. But

$$\left.\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}\right)\right|_{x=y=\frac{1}{2}} \;=\; -3 \cdot 9 - 3^2 \;<\; 0 \quad \Longrightarrow \quad \text{saddle point}\,.$$

4) Answer: d)

In both cases, along the two horizontal line segments in C, $\mathbf{T}$ is a multiple (depending on the particular parametrization) of $\mathbf{i}$, which is perpendicular to $\mathbf{F}$ and $\mathbf{G}$. In both cases, then, the contributions of the two horizontal sides to the integral vanish. On the vertical line segments, $\mathbf{F}(\pm 1, t) = f(1)\mathbf{j}$ and $\mathbf{G}(\pm 1, t) = \pm f(1)\mathbf{j}$ and, with the natural linear parametrization, $\mathbf{T} = \mathbf{j}$ on the right vertical line segment, $\mathbf{T} = -\mathbf{j}$ on the left vertical line segment. Thus, in the case of the integral I, the two vertical contributions cancel, whereas in the case of J the vertical contributions are both strictly positive.

5) Answer: b)

A vector field on the plane cannot be conservative unless its two-dimensional curl vanishes. Since

$$\operatorname{curl} \mathbf{G}(x, y) \;=\; \frac{\partial}{\partial x} e^y - \frac{\partial}{\partial y}(2y) \;=\; -2\,,$$

$\mathbf{G}$ is not conservative. One easily sees that the two-dimensional curl of $\mathbf{F}$ does vanish, and that

$$\operatorname{grad}(e^{xy} + e^y) \;=\; \left(\frac{\partial}{\partial x}(e^{xy} + e^y)\right)\mathbf{i} + \left(\frac{\partial}{\partial y}(e^{xy} + e^y)\right)\mathbf{j} \;=\; \mathbf{F}\,,$$

exhibiting $f(x, y) = e^{xy} + e^y$ as a potential function for $\mathbf{F}$.

2 Part II

1a) In terms of spherical coordinates, the cone $z = \sqrt{x^2 + y^2}$ is described by the equation $\phi = \frac{\pi}{4}$, the x-y plane by $\phi = \frac{\pi}{2}$, and the sphere of radius 2 by $\rho = 2$; a point lies to the right of the y-z plane precisely when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Thus R corresponds to the region

$$0 \le \rho \le 2\,, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\,, \quad \frac{\pi}{4} \le \phi \le \frac{\pi}{2}$$

in ρ-θ-ϕ space. Recall that $z = \rho\cos\phi$, and $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$. The integration can be performed in any order without affecting the limits:

$$\int_R z\,dV \;=\; \int_{\phi=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\rho=0}^{2} \rho^3 \sin\phi\, \cos\phi\, d\rho\, d\theta\, d\phi\,.$$

b) In terms of cylindrical coordinates, the cone is $z = r$, and the sphere of radius 2 is $r = \sqrt{4 - z^2}$. These two surfaces intersect when $z = r = \sqrt{2}$. For any particular value of z between 0 and $\sqrt{2}$, the region inside the sphere, below the cone, is characterized by the inequalities $z \le r \le \sqrt{4 - z^2}$. The inequality $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ describes the half-space to the right of the y-z plane, just as in the case of spherical coordinates. Conclusion: R corresponds to the region

$$0 \le z \le \sqrt{2}\,, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\,, \quad z \le r \le \sqrt{4 - z^2}$$

in r-θ-z space. Since $dV = r\,dr\,d\theta\,dz$,

$$\int_R z\,dV \;=\; \int_{z=0}^{\sqrt{2}} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=z}^{\sqrt{4-z^2}} r\,z\,dr\,d\theta\,dz\,.$$

The θ-integration can be shifted to the left or the right without affecting the limits, but interchanging the order of the other two variables requires splitting up the region of integration into two sub-regions, one inside the cylinder $r = \sqrt{2}$, the other outside it.

c) The simplest way to do this is to write I as a difference of two separate integrals (note that a difference of two integrals is also a sum of integrals – just reverse the sign of the second integral). As in b), the region R lies between the horizontal planes $z = 0$ and $z = \sqrt{2}$. For any z in this range, the horizontal plane at height z cuts R in the semi-ring

$$x \geq 0, \quad z^2 \leq x^2 + y^2 \leq 4 - z^2,$$

which can be viewed as the semi-disc $x \geq 0$, $x^2 + y^2 \leq 4 - z^2$, from which the semi-disc $x \geq 0$, $x^2 + y^2 \leq z^2$ has been removed. Thus

$$\begin{aligned}\int_R z\,dV &= \int_{z=0}^{\sqrt{2}} \int_{y=-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{x=0}^{\sqrt{4-z^2-y^2}} z\,dx\,dy\,dz \\ &\quad - \int_{z=0}^{\sqrt{2}} \int_{y=-z}^{z} \int_{x=0}^{\sqrt{z^2-y^2}} z\,dx\,dy\,dz\end{aligned}$$

A similar alternative is to describe the region R as the quarter-sphere $x \geq 0$, $0 \leq z \leq \sqrt{4 - x^2 - y^2}$, from which we have removed the "half ice-cream cone" $x \geq 0$, $\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2}$

$$\begin{aligned}\int_R z\,dV &= \int_{x=0}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{\sqrt{4-x^2-y^2}} z\,dz\,dy\,dx \\ &\quad - \int_{x=0}^{\sqrt{2}} \int_{y=-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{z=\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z\,dz\,dy\,dx.\end{aligned}$$

d) Beginning with the identity in a), we find

$$\begin{aligned}\int_R z\,dV &= \int_{\phi=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\rho=0}^{2} \rho^3 \sin\phi \cos\phi \,d\rho\,d\theta\,d\phi \\ &= \left(\int_0^2 \rho^3\,d\rho\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\right)\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \cos\phi\,d\phi\right) \\ &= 4 \cdot \pi \cdot \frac{1}{2}(1 - \frac{1}{2}) \;=\; \pi\end{aligned}$$

2) A closest point exists for geometric reasons. We apply the Lagrange multiplier method to

$$f(x,y,z) \;=\; (\text{distance})^2 \;=\; (x-2)^2+(y-3)^2+z^2$$

and the constraint function $g(x,y,z) = x^2+2y^2-z^2$. Note that

$$\operatorname{grad} g \;=\; 2x\,\mathbf{i}+4y\,\mathbf{j}-2z\,\mathbf{k}$$

vanishes at the origin, but no place else. Thus, to find the closest point(s), we need to look at the origin in addition to the solution(s) of

$$\operatorname{grad} f \;=\; \lambda\,\operatorname{grad} g\,, \qquad g \;=\; 0\,.$$

Written out explicitly, these equations become

$$2(x-2) \;=\; 2\lambda x\,,\quad 2(y-3) \;=\; 4\lambda y\,,\quad 2z \;=\; -2\lambda z\,,\quad x^2+2y^2 \;=\; z^2\,.$$

The only point on the cone where $z=0$ is the origin, which needs to be dealt with anyhow. Thus we can divide the third equation by z, and conclude $\lambda=-1$. Now $x=1$ from the first equation, $y=1$ from the second, and finally $z=\pm\sqrt{3}$ from the constraint. The restriction $z\geq 0$ eliminates $z=-\sqrt{3}$, leaving only the point $(1,1,\sqrt{3})$ and the origin as potential answers. Comparing values, one can exclude the origin. Answer: $(1,1,\sqrt{3})$ is the (unique) closest point.

3a) Parametrize C linearly: $x = x_1 + t(x_2 - x_1)$, $y = y_1 + t(y_2 - y_1)$, $0 \le t \le 1$. Then

$$\begin{aligned}
\int_C x\,dy &= \int_0^1 (x_1 + t(x_2 - x_1))(y_2 - y_1)dt \\
&= (x_1 + \frac{1}{2}(x_2 - x_1))(y_2 - y_1) \;=\; \frac{1}{2}(x_2 + x_1)(y_2 - y_1) \\
&= \frac{1}{2}(x_1 y_2 - x_2 y_1) \;-\; \frac{1}{2}\,x_1 y_1 \;+\; \frac{1}{2}\,x_2 y_2\,.
\end{aligned}$$

b) Green's theorem asserts that $\int_{\partial P} M\,dx + N\,dy = \int_P (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dx\,dy$. In this particular case,

$$\int_{\partial P} x\,dy \;=\; \int_P 1 \cdot dx\,dy \;=\; \text{area}(P)\,.$$

c) The boundary of P consists of five straight line segments. Applying a) to each of the five segments and using b), one finds

$$\begin{aligned}
\text{area}(P) &= \int_{\partial P} x\,dy = \\
&= \frac{1}{2}(x_1 y_2 - x_2 y_1) \;-\; \frac{1}{2}\,x_1 y_1 \;+\; \frac{1}{2}\,x_2 y_2 \\
&\quad + \frac{1}{2}(x_2 y_3 - x_3 y_2) \;-\; \frac{1}{2}\,x_2 y_2 \;+\; \frac{1}{2}\,x_3 y_3 \\
&\quad + \frac{1}{2}(x_3 y_4 - x_4 y_3) \;-\; \frac{1}{2}\,x_3 y_3 \;+\; \frac{1}{2}\,x_4 y_4 \\
&\quad + \frac{1}{2}(x_4 y_5 - x_5 y_4) \;-\; \frac{1}{2}\,x_4 y_4 \;+\; \frac{1}{2}\,x_5 y_5 \\
&\quad + \frac{1}{2}(x_5 y_1 - x_1 y_5) \;-\; \frac{1}{2}\,x_5 y_5 \;+\; \frac{1}{2}\,x_1 y_1\,.
\end{aligned}$$

With the obvious cancellation, this gives the answer as

$$\begin{aligned}
\text{area}(P) &= \frac{1}{2}(x_1 y_2 + x_2 y_3 + x_3 y_4 + x_4 y_5 + x_5 y_1) \\
&\quad - \frac{1}{2}(x_2 y_1 + x_3 y_2 + x_4 y_3 + x_5 y_4 + x_1 y_5)\,.
\end{aligned}$$