Math 21a Review Sheet

This sheet is an attempt to organize the most important ideas so far in Math 21a to help you review for the first midterm exam. It is in no way a complete summary of the course so far, and anything covered so far could appear on the exam, whether or not it is mentioned here. You should understand everything in more detail than is given here (refer to the textbook).

Vectors and Vector Algebra

Be sure you understand what vectors and scalars are, how to measure length, how to carry out addition and scalar multiplication, how to express a vector in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, etc.

The dot product is defined by $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$ where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. The dot product of two vectors is a scalar. It can be calculated by

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = a_1b_1 + a_2b_2 + a_3b_3.$$

All the algebraic identities one would hope for hold (e.g., $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$, or $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$). Vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular iff $\mathbf{A} \cdot \mathbf{B} = 0$. We have $|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A}$.

The vector projection of $\mathbf{B}$ onto $\mathbf{A}$ can be calculated using the dot product, via

$$\mathrm{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{B} \cdot \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}}\right) \mathbf{A}.$$

The cross product $\mathbf{A} \times \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is defined to have length $|\mathbf{A}||\mathbf{B}| \sin\theta$ (θ is the angle, as above), and to be orthogonal to $\mathbf{A}$ and $\mathbf{B}$ in the direction given by the right-hand rule. The cross product of two vectors is a vector. It can be calculated by

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

This is best remembered using a determinant (see equation (9) on page 818 of the textbook). Few of the algebraic identities one would hope for hold. The cross product is anticommutative ($\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$). Vectors $\mathbf{A}$ and $\mathbf{B}$ are parallel iff $\mathbf{A} \times \mathbf{B} = \mathbf{0}$. More generally, $|\mathbf{A} \times \mathbf{B}|$ is the area of the parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$.

The absolute value of the triple scalar product $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ is the volume of the parallelepiped spanned by $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and its sign depends on the order in which those vectors occur.

Lines and Planes in Space

Be sure you know everything in Section 10.5 (and why it works).

The line through (x_0, y_0, z_0) parallel to $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is parametrized by $x = x_0 + tA$, $y = y_0 + tB$, $z = z_0 + tC$, with $-\infty < t < \infty$.

The distance from a point S to a line through P parallel to $\mathbf{v}$ is given by

$$\frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$

The plane through (x_0, y_0, z_0) normal to $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is given by $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$, which can be written as $Ax + By + Cz = D$ if we set $D = Ax_0 + By_0 + Cz_0$.

The distance from a point S to a plane containing a point P and normal to a vector $\mathbf{n}$ is the length of $\mathrm{proj}_{\mathbf{n}}\vec{PS}$, i.e., is given by

$$\left|\vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|.$$

To find the angle between two planes, use the dot product to find the angle between their normal vectors. One way to find a vector parallel to the line of intersection of two planes is to take the cross product of their normal vectors.

Vector-valued Functions and Space Curves

You should know what a vector-valued function is, and understand why most operations (limits, derivatives, integrals) can be carried out by components.

As far as new formulas go, it is important to remember that if $\mathbf{A}$ and $\mathbf{B}$ are vector-valued functions of t, then

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$$

and

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}.$$

(Be careful about the order of the factors in the cross products.)

If $\mathbf{u}$ is a (differentiable) vector-valued function of t such that $|u|$ is constant, then

$$\mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = 0.$$

Projectile Motion; Planetary Motion

You don't need to know much at all about physics, but you should understand how one solves and interprets the solutions of the differential equation in Section 11.2, and you should understand the derivation of Kepler's second law (that a planet sweep out equal area in equal times).

Arc Length

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ parametrizes a curve as t goes from a to b, then the arc length is

$$\int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt.$$

An "arc length parameter" parametrizes a curve according to arc length. If the starting point is $\mathbf{r}(t_0)$ and the parametrization goes in the same direction as $\mathbf{r}$, then it is given by the formula

$$s(t) = \int_{t_0}^t |\mathbf{r}'(\tau)|\,d\tau.$$

Expressing $\mathbf{r}$ as a function of s (rather than t) often gives a more natural parametrization.

Multivariable Functions and Partial Derivatives

You should know what a function of several variables is and how to graph one. Furthermore, you should know what level curves or surfaces are and how to interpret them. You should be able to compute partial derivatives (both directly and via implicit differentiation), and familiar with different sorts of notation for them, so equations like

$$\frac{\partial^4 f}{\partial y\,\partial x\,\partial x\,\partial z} = f_{zxxy}$$

are familiar.

The main fact about partial derivatives that isn't familiar from single-variable calculus is the equality of mixed partial derivatives: if $f(x,y)$ is a decent function (in particular, its mixed 2nd order partial derivatives f_{xy} and f_{yx} exist and are continuous), then $f_{xy} = f_{yx}$, i.e.,

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}.$$

Linearization and Differentials

If $f(x, y)$ is a differentiable function of two variables, then near a point (x_0, y_0) the approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

holds. (In fact, to say f is differentiable means that such an approximation holds, with error $\varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$, where $\varepsilon_1, \varepsilon_2 \to 0$ as $x, y \to x_0, y_0$.) The right hand side of this approximation is known as the linearization of f, since f is being approximated by a linear function.

One can bound the error in the approximation as follows. If f has continuous 2nd order partial derivatives in a rectangle R centered at (x_0, y_0), and if M is any upper bound on the values of $|f_{xx}|$, $|f_{xy}|$ and $|f_{yy}|$ on R, then the error involved in linearizing f is at most

$$\frac{M}{2}\left(|x - x_0| + |y - y_0|\right)^2 .$$

You should be familiar with using differentials to estimate change: moving from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$ changes f by approximately

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

You should be able to use this to estimate both absolute and relative changes.

You should also know that all these results also hold when there are more than two variables.

The Chain Rule

If $w = f(x, y)$ is a differentiable function of x and y, and x and y are differentiable functions of t, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

The analogous result holds if f is a function of three or more variables: if $w = f(x, y, z)$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Similarly, if $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

The pattern is that if w is a function of r through several intermediate variables, apply the ordinary chain rule to compute $\partial w/\partial r$ by going through each intermediate variable, and then add the results.

You should be able to apply the chain rule directly or in the context of implicit differentiation.