1. MIDTERM 2 – MATH 21A

Standard disclaimer: this is not a complete summary; anything covered in the course could show up in the exam, whether or not it is mentioned here; you should understand everything in more detail than what is given here (refer to the textbook).

1.1. MULTIVARIABLE FUNCTIONS AND PARTIAL DERIVATIVES (12.1, 12.3, 12.4, 12.5, 12.7)

- Partial derivatives: what do they mean? How to compute them?
- Tangent plane. If $z = f(x, y)$, the tangent plane to the graph at (a, b) is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Differential

$$df = f_x\, dx + f_y\, dy$$

- Directional derivatives (rate of change of f in a direction $\mathbf{v}$). What do they mean? If $\mathbf{v}$ is an unit vector, then:

$$f_{\mathbf{v}} = grad\, f \cdot v$$

where

$$grad\, f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

- Meaning of the gradient vector:

$$\begin{aligned} &grad\, f \text{ points in the direction of maximum increase of } f \\ &\|grad\, f\| \text{ is the rate of change of } f \text{ in that direction} \end{aligned}$$

- Properties

$$\begin{aligned} &grad\, f \text{ is perpendicular to the level curve/surface of } f \\ &grad\, f \text{ is larger when the level curves/surfaces are closer} \end{aligned}$$

- Chain rule example: if $z = f(x, y)$ and $x = g(u, v)$, $y = h(u, v)$, then

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\end{aligned}$$

- Second order partials: f_{xx}, f_{yy}, and $f_{xy}(= f_{yx}$ "most of the time").

1.2. OPTIMIZATION (12.8, 12.9)

- CRITICAL POINTS

$$grad\, f = \mathbf{0} \text{ or } grad\, f \text{ is undefined}$$

- LOCAL MINIMA, LOCAL MAXIMA and SADDLE POINTS (= the other critical points).
- The second derivative test: if $P(x_0, y_0)$ is a critical point, calculate $D = f_{xx}f_{yy} - f_{xy}^2$ at P. Then

$$\begin{aligned} D &< 0 \Rightarrow P \text{ is a saddle point}\\ D &> 0 \text{ and } f_{xx}(P) > 0 \Rightarrow P \text{ is a local minimum}\\ D &> 0 \text{ and } f_{xx}(P) < 0 \Rightarrow P \text{ is a local maximum}\\ D &= 0 \Rightarrow \text{anything can happen}\end{aligned}$$

- Constrained optimization (Lagrange multipliers): in order to optimize f subject to the constraint $g = c$, solve

$$\begin{cases} grad\, f = \lambda\, grad\, g \\ g = c \end{cases} \text{ or } \begin{cases} grad\, g = \mathbf{0} \\ g = c \end{cases}$$

  Most of the time the second system yields no solutions, but you must check it anyway.

- Let a function $f(x, y)$ be defined on a closed bounded set D. Then f MUST assume minimum and maximum values in that set D. Those can occur either at interior points (easily found as interior critical points) or at boundary points of D (usually found through Lagrange multipliers).

1.3. MULTIPLE INTEGRALS (13.1-13.6)

- $$\begin{aligned} \int\int_R f\,dA \quad &\Rightarrow \quad \text{Double Integral} \\ \int\int\int_W f\,dV \quad &\Rightarrow \quad \text{Triple Integral} \end{aligned}$$

- If f is a density function, the multiple integral above gives you total mass.
- If $f = 1$, we get the Area of R (Double Integral) or the Volume of W (Triple Integral)
- Divide the double (triple) integral by the area of R (volume of W) to get average value of f over R (W).
- Double integrals gives us the signed volume under the graph of $f(x, y)$.
- If δ is a density function (mass per area), you can get the coordinates of the center of mass of R

  $$\bar{x} = \frac{\int\int_R x\delta\,dA}{\int\int_R \delta\,dA};\ \bar{y} = \frac{\int\int_R y\delta\,dA}{\int\int_R \delta\,dA}$$

  or, if δ is mass per volume

  $$\bar{x} = \frac{\int\int\int_W x\delta\,dA}{\int\int\int_W \delta\,dA};\ \bar{y} = \frac{\int\int\int_W y\delta\,dA}{\int\int\int_W \delta\,dA};\ \bar{z} = \frac{\int\int\int_W z\delta\,dA}{\int\int\int_W \delta\,dA}$$

- How to calculate them? Use iterated integrals! Double integrals look like:

  $$\int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)} f(x, y)\,dy\,dx$$

  Note that a_1 and a_2 must be constants, while b_1 and b_2 might depend on the exterior variable (in this case, x).

  Triple integrals look like

  $$\int_{a_1}^{a_2}\int_{b_1(x)}^{b_2(x)}\int_{c_1(x,y)}^{c_2(x,y)} f(x, y, z)\,dz\,dy\,dx$$

- The order of integration might be reversed. In this case, sketch the region of integration and get the new bounds from the picture. The exterior bounds will always be constants!

- POLAR COORDINATES

$$x = r\cos\theta;\ y = r\sin\theta \iff r = \sqrt{x^2+y^2};\ \tan\theta = \frac{y}{x}$$

If the region R is easily described in polar coordinates, use $dA = r\,dr\,d\theta$:

$$\int\int_R f(x,y)\,dx\,dy = \int\int_R f(r,\theta)\,r\,dr\,d\theta$$

- CYLINDRICAL COORDINATES

$$x = r\cos\theta;\ y = r\sin\theta;\ z = z \iff r = \sqrt{x^2+y^2};\ \tan\theta = \frac{y}{x};\ z = z$$

If the region W is easily described in cylindrical coordinates, use $dV = r\,dr\,d\theta\,dz$:

$$\int\int\int_W f(x,y,z)\,dx\,dy\,dz = \int\int\int_W f(r,\theta,z)\,r\,dr\,d\theta\,dz$$

- SPHERICAL COORDINATES

$$\left.\begin{array}{l} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{array}\right\} \iff \left\{\begin{array}{r} \rho = \sqrt{x^2+y^2+z^2} \\ \tan\theta = \frac{y}{x} \\ \tan\phi = \frac{\sqrt{x^2+y^2}}{z} \end{array}\right.$$

where we take $0 \le \rho$; $0 \le \phi \le \pi$; $0 \le \theta \le 2\pi$. If the region W is easily described in spherical coordinates, use $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$:

$$\int\int\int_W f(x,y,z)\,dx\,dy\,dz = \int\int\int_W f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

1.4. VECTOR FIELDS (14.2)

- Vector Fields associate a vector to each point of the plane or 3D-space. An example in 3D is

$$\mathbf{F}(x,y,z) = \sin(xz)\mathbf{i} + (y^2 - z^2)\mathbf{j} + 75\mathbf{k}$$

In general, they look like

$$\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$$

- GRADIENT VECTOR FIELDS are gradient of SOME FUNCTION. For example

$$\begin{aligned} \mathbf{F}(x,y,z) &= z\cos(xz)\mathbf{i} + 2y\mathbf{j} + (x\cos(xz) - 2z)\mathbf{k} \\ \text{is the gradient of } f(x,y,z) &= \sin(xz) + y^2 - z^2 \end{aligned}$$

  If $\mathbf{F} = grad\, f$ we say that f is a POTENTIAL FUNCTION for $\mathbf{F}$.