1 Math 21a, Fall 1998
Sample Midterm 2

This practice exam is the actual second midterm of math 21a as taught in the spring of 1996. You may want to take this practice exam under exam-like conditions.

The exam was scheduled earlier in the semester than our second midterm. For that reason, it does not cover line integrals and Green's theorem, which *will* be covered on our second midterm. To check your familiarity with this material, you should also work out the answers to the "additional questions on line integrals" elsewhere on the web site.

The students in 1996 had two hours to complete the exam. Here are the instructions they were given:

"The exam has two parts. Part I consists of eleven multiple choice questions; questions 1 – 5 are worth 3 points each, and questions 6 – 11 are each worth 5 points. Part II consists of four longer problems; problem 1 is worth 10 points, and problems 2 – 4 are worth 15 points each. Please answer the questions in the spaces provided for this purpose. In part II, you must justify your answers."

2 Part I (45 points)

Multiple choice. Questions 1 – 5 count 3 points each, questions 6 – 11 count 5 points each. Every question has only one correct answer – circle it!

Questions 1-7 refer to the following three regions in space:
D_1 is the region bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the xy-plane;
D_2 is the region bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $z = 1$
D_3 is the region lying between the two spheres $x^2+y^2+z^2 = 1$, $x^2+y^2+z^2 = 4$, and above the xy-plane.

Note: each of these regions may come up as correct answer once, more than once, or not at all.

1) The triple integral $\int_0^{2\pi} \int_1^2 \int_0^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$ computes the volume of:

a) D_1 ; c) D_3 ;
b) D_2 ; d) none of these.

2) The triple integral $\int_0^{2\pi} \int_1^2 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$ computes the volume of:

a) D_1 ; c) D_3 ;
b) D_2 ; d) none of these.

3) The triple integral $\int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$ computes the volume of:

a) D_1 ; c) D_3 ;
b) D_2 ; d) none of these.

4) The triple integral $\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$ computes the volume of:

a) D_1 ; c) D_3 ;
b) D_2 ; d) none of these.

5) The triple integral $\int_0^{\sqrt{3}} \int_0^{2\pi} \int_1^{\sqrt{4-r^2}} r \, dz \, d\theta \, dr$ computes the volume of:

a) D_1 ; c) D_3 ;
b) D_2 ; d) none of these.

6) Denote the integral of z^2 over D_1, D_2, D_3 by, respectively, I_1, I_2, I_3. Then:

a) $I_1 < I_2 < I_3$;

b) $I_1 < I_3 < I_2$;

c) $I_2 < I_1 < I_3$;

d) $I_2 < I_3 < I_1$;

e) $I_3 < I_1 < I_2$;

f) $I_3 < I_2 < I_1$.

Important: you do not need to calculate the integrals.

7) Denote the average value of z^2 over D_1, D_2, D_3 by, respectively, A_1, A_2, A_3. Then:

a) $A_1 < A_2 < A_3$;

b) $A_1 < A_3 < A_2$;

c) $A_2 < A_1 < A_3$;

d) $A_2 < A_3 < A_1$;

e) $A_3 < A_1 < A_2$;

f) $A_3 < A_2 < A_1$.

Important: you do not need to calculate the average values.

8) Reversing the order of integration transforms $\int_{-2}^{0}\int_{2x+4}^{4-x^2} f(x,y)\,dy\,dx$ into which of the following integrals:

a) $\int_{2x+4}^{4-x^2}\int_{-2}^{0} f(x,y)\,dx\,dy$;

b) $\int_{0}^{4}\int_{\sqrt{4-y}}^{(y-4)/2} f(x,y)\,dx\,dy$;

c) $\int_{0}^{4}\int_{-2}^{0} f(x,y)\,dx\,dy$;

d) $\int_{0}^{4}\int_{(y-4)/2}^{\sqrt{4-y}} f(x,y)\,dx\,dy$;

e) $\int_{-2}^{0}\int_{2y+4}^{4-y^2} f(x,y)\,dx\,dy$;

f) $\int_{0}^{4}\int_{-\sqrt{4-y}}^{(y-4)/2} f(x,y)\,dx\,dy$;

9) The potential maxima and minima of the function $f(x,y) = x - 2y + 5$ on the ellipse $2x^2 + 8y^2 = 1$ (as predicted by the Lagrange multiplier method) occur at the following collection of points:

a) $(1,-2)$ and $(-1,2)$;

b) $(2,-1)$ and $(-2,1)$;

c) $(\frac{1}{4},-\frac{1}{2})$ and $(-\frac{1}{4},\frac{1}{2})$;

d) $(\frac{1}{4},\frac{1}{2})$, $(-\frac{1}{4},\frac{1}{2})$, $(\frac{1}{4},-\frac{1}{2})$, $(-\frac{1}{4},-\frac{1}{2})$;

e) $(\frac{1}{2},-\frac{1}{4})$ and $(-\frac{1}{2},\frac{1}{4})$;

f) $(\frac{1}{2},\frac{1}{4})$, $(-\frac{1}{2},\frac{1}{4})$, $(\frac{1}{2},-\frac{1}{4})$, $(-\frac{1}{2},-\frac{1}{4})$.

10) The extrema of the function $f(x,y) = x^4 - 32x + y^2 + y$ on the disc $x^2 + y^2 \leq 1$ occur as follows:

a) both the maximum and the minimum in the interior of the disc;
b) the maximum in the interior and the minimum on the boundary;
c) the minimum in the interior and the maximum on the boundary;
d) both the maximum and the minimum on the boundary of the disc;
e) f does not assume a global maximum on the disc;
f) f does not assume a global minimum on the disc.

11) Which statement about the maxima and minima of the function

$$f(x,y) = \frac{5}{1 + \sqrt{x^2 + y^2}}$$

on the disc $x^2 + y^2 \leq 9$ is correct?

a) the minimum occurs at a unique point and the maximum occurs at a unique point;
b) the minimum occurs at a unique point but the maximum is assumed at more than one point;
c) the minimum is assumed at more than one point but the maximum occurs at a unique point;
d) the minimum is assumed at more than one point and the maximum is assumed at more than one point;
e) there is no minimum but the maximum is assumed at more than one point;
f) the minimum is assumed at more than one point but there is no maximum.

3 Part II (55 points)

Problem 1 is worth 10 points, problems 2 – 4 are worth 15 points each. You should attempt all parts of all problems. Show your work!

1a) Find the tangent plane to the graph of the function $f(x, y) = x\,y^2$ at the point $(2, -1, 2)$.

1b) Give a parametric description of the line through the point $(2, -1, 2)$ and normal to the graph of the function $f(x, y) = x\,y^2$.

1c) Find a non-zero vector in the xy-plane normal to the level curve $f(x, y) = 2$ of the function $f(x, y) = x\,y^2$ at the point $(2, -1)$.

2a) Find and classify (as local minimum, local maximum, or saddle point) all the critical points of the function $f(x, y) = x^4 + y^4 - 4xy + 4$.

2b) Does the function $f(x, y)$ of part (a) have a global maximum? If yes, where? If no, why?

2c) Does the function $f(x, y)$ of part (a) have a global minimum? If yes, where? If no, why?

3) Let S be the surface described by the equation $xyz^2 = 32$ in the first octant (i.e., with $x > 0$, $y > 0$, $z > 0$). Find the point on S that is closest to the origin.

4) Let D be the solid bounded above by the paraboloid $z = 18 - x^2 - y^2$, below by the paraboloid $z = x^2 + y^2$, and lying outside the cylinder $x^2 + y^2 = 1$. The density of this solid varies with the distance from the z-axis: $\delta(x, y, z) = 1/\sqrt{x^2 + y^2}$. Find its mass!