1 Planetary Motion

Let $\mathbf{X} = \mathbf{X}(t)$ be the position vector of a particle of mass m moving under the influence of an inverse-square force $\mathbf{F} = -cm\,|\mathbf{X}|^{-3}\,\mathbf{X}$. Then

(1) $$\mathbf{X}'' = -c\,|\mathbf{X}|^{-3}\,\mathbf{X}.$$

The vector $\mathbf{L} =_{\text{def}} \mathbf{X} \times \mathbf{X}'$ is time independent because

(2) $$\frac{d}{dt}\mathbf{L} = \mathbf{X}' \times \mathbf{X}' + \mathbf{X} \times \mathbf{X}'' = -\frac{c}{|\mathbf{X}|^3}\mathbf{X} \times \mathbf{X} = \mathbf{0}.$$

Physically, (2) asserts the constancy of angular momentum. The vector $\mathbf{L}$ is perpendicular to $\mathbf{X}$. Hence, if $\mathbf{L} \neq \mathbf{0}$, the particle moves in the plane normal to $\mathbf{L}$. On the other hand, if $\mathbf{L} = \mathbf{0}$, it is not so difficult to see that the particle moves along a straight line through the origin. Let us exclude this degenerate case. We can then rotate the coordinate system so that $\mathbf{L}$ is a positive multiple of the third standard unit vector $\mathbf{k}$, say $\mathbf{L} = \ell\mathbf{k}$ with $\ell > 0$. This has the effect of making the particle move in the x-y plane. From the definition of the cross product, we get

$$\begin{aligned} \ell &= |\mathbf{X}| \times (\text{length of the component of the velocity normal to } \mathbf{X}) \\ (3) \quad &= |\mathbf{X}|^2 \times \pm\,(\text{angular velocity of the particle as seen from the origin}) \\ &= 2 \times (\text{the rate at which the vector } \mathbf{X} \text{ sweeps out area})\,. \end{aligned}$$

That is Kepler's second law: the position vector sweeps out area at a constant rate.

We now calculate the time derivative of the cross product of $\mathbf{L}$ and the unit vector in the direction of $\mathbf{X}$; this calculation depends on the triple cross product identity

(4) $$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$$

which holds for any three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. Since $\mathbf{L}$ is constant,

$$\begin{aligned} \frac{d}{dt}\left(\mathbf{L} \times \frac{\mathbf{X}}{|\mathbf{X}|}\right) &= \mathbf{L} \times \frac{d}{dt}\left(\frac{\mathbf{X}}{|\mathbf{X}|}\right) = (\mathbf{X} \times \mathbf{X}') \times \left(\frac{\mathbf{X}'}{|\mathbf{X}|} - \frac{\mathbf{X} \cdot \mathbf{X}'}{|\mathbf{X}|^3}\mathbf{X}\right) \\ (5) \quad &= \frac{\mathbf{X} \cdot \mathbf{X}'}{|\mathbf{X}|}\mathbf{X}' - \frac{\mathbf{X}' \cdot \mathbf{X}'}{|\mathbf{X}|}\mathbf{X} - \frac{\mathbf{X} \cdot \mathbf{X}'}{|\mathbf{X}|}\mathbf{X}' + \frac{(\mathbf{X} \cdot \mathbf{X}')^2}{|\mathbf{X}|^3}\mathbf{X} \\ &= \frac{(\mathbf{X} \cdot \mathbf{X}')^2}{|\mathbf{X}|^3}\mathbf{X} - \frac{|\mathbf{X}'|^2}{|\mathbf{X}|}\mathbf{X}. \end{aligned}$$

Let ϕ denote the angle between $\mathbf{X}$ and $\mathbf{X}'$. Then

(6)
$$\ell^2 = |\mathbf{X}|^2|\mathbf{X}'|^2(\sin\phi)^2 = |\mathbf{X}|^2|\mathbf{X}'|^2 - |\mathbf{X}|^2|\mathbf{X}'|^2(\cos\phi)^2 = |\mathbf{X}|^2|\mathbf{X}'|^2 - (\mathbf{X} \cdot \mathbf{X}')^2$$

which allows us to re-write (5) as follows:

$$\frac{d}{dt}\left(\mathbf{L}\times\frac{\mathbf{X}}{|\mathbf{X}|}\right) = -\frac{\ell^2}{|\mathbf{X}|^3}\mathbf{X} = \frac{\ell^2}{c}\mathbf{X}'' = \frac{d}{dt}\left(\frac{\ell^2}{c}\mathbf{X}'\right). \tag{7}$$

It follows that

$$\mathbf{E} =_{\mathrm{def}} \mathbf{L}\times\frac{\mathbf{X}}{|\mathbf{X}|} - \frac{\ell^2}{c}\mathbf{X}' \tag{8}$$

is a constant vector. This vector is normal to $\mathbf{L} = \ell\mathbf{k}$ since both of the summands are. If $\mathbf{E}$ vanishes, one can show readily that the particle moves along a circle centered at the origin - at a constant speed because of Kepler's second law. If $\mathbf{E} \neq \mathbf{0}$, we can rotate the coordinate system around the z axis so that $\mathbf{E}$ becomes a negative multiple of $\mathbf{j}$, say $\mathbf{E} = -\varepsilon\mathbf{j}$ with $\varepsilon > 0$. Recall that motion takes place in the x-y plane. We can therefore describe the position vector $\mathbf{X}$ in terms of polar coordinates: $\mathbf{X} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$. Then $r = |\mathbf{X}|$ is the length of the position vector, and

$$\mathbf{X}' = r'(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) + r\theta'(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}). \tag{9}$$

Hence (8) can be re-written as follows:

$$\begin{aligned} -\varepsilon\mathbf{j} &= \\ &= \ell\mathbf{k}\times(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) - \frac{\ell^2}{c}r'(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) + \frac{\ell^2 r}{c}\theta'(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j}) \qquad (10) \\ &= \left(\frac{\ell^2 r}{c}\theta' - \ell\right)(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j}) - \frac{\ell^2}{c}r'(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}). \end{aligned}$$

We take the dot product with the vector $c\ell^{-1}(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$, to conclude

$$\frac{c\varepsilon}{\ell}\cos\theta = \ell r\theta' - c. \tag{11}$$

Up to sign, θ' specifies the angular velocity of the particle, so (3) translates into the formula $\theta' = \pm\ell r^{-2}$. To pin down the sign, substitute the polar description of $\mathbf{X}$ and the equation (9) into the definition $\mathbf{L} = \mathbf{X}\times\mathbf{X}'$ of the vector $\mathbf{L} = \ell\mathbf{k}$. A short calculation gives $\theta' = \ell r^{-2}$. We use this formula to eliminate θ' from (11). The resulting equation

$$r\left(1 + \frac{\varepsilon}{\ell}\cos\theta\right) = \frac{\ell^2}{c} \tag{12}$$

describes a conic section with axis along the x axis and one focus at the origin. We have verified Kepler's first law: a particle subject to an inverse-square force, as in (1), moves along a conic section.