1 Math 21a, Fall 1998
Sample Final Exam

(May 17, 1996; Some Biochem questions from Fall 1997)

The exam has two parts. Part I consists of seven questions, some multiple choice, some questions with short answers. Part I consists of seven questions, some multiple choice, some questions with short answers. Problem 7 counts twelve points, the others six points each. Part II consists of seven longer problems. Please answer the questions in the spaces provided for this purpose. In part II, you must justify your answers.

Good luck!

Your name:

2 Part I (48 points)

Multiple choice/short questions – no justification is required. Each multiple choice question has only one correct answer.

1) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are vectors in the plane as shown below. Note that $|\mathbf{A}| = |\mathbf{C}|$ and $|\mathbf{B}| = |\mathbf{D}|$.

In each case, circle the correct choice.

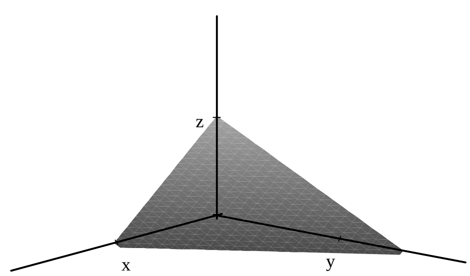
a) Which is greater?	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{C} \cdot \mathbf{D}$
b) Which is greater?	$\mathbf{A} \cdot \mathbf{B}$	$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})$
c) Which has greater length?	$\mathbf{A} \times \mathbf{B}$	$\mathbf{C} \times \mathbf{D}$
d) Which has greater length?	$\mathbf{C} + \mathbf{D}$	$\mathbf{C} - \mathbf{D}$
e) Which is zero?	$(\mathbf{A} \times \mathbf{B}) \times \mathbf{B}$	$\mathbf{A} \times (\mathbf{B} \times \mathbf{B})$
f) Which points up and out of the page?	$\mathbf{A} \times \mathbf{B}$	$\mathbf{D} \times \mathbf{C}$

2) All but one of the following parametrizations, with $-\infty < t < \infty$, trace out the same path in three dimensional Euclidean space. Circle the one that is different.

a) $x = t\,,\ y = t^2\,,\ z = t^3\,;$
b) $x = -t\,,\ y = -t^2\,,\ z = -t^3\,;$
c) $x = -t\,,\ y = t^2\,,\ z = -t^3\,;$
d) $x = t - 1\,,\ y = (t-1)^2\,,\ z = (t-1)^3\,;$
e) $x = 2t\,,\ y = 4t^2\,,\ z = 8t^3\,.$

3) Only one of the following planes "cuts the corner off" the first octant. Which one?

a) $3x - y + 2z = 6\,;$
b) $3x + y + 2z = -6\,;$
c) $-3x - y + 2z = 6\,;$
d) $3x + y + 2z = 6\,;$
e) $3x + y - 2z = -6$



4) Give an example of a vector field $\mathbf{F} = \mathbf{F}(x, y)$ in the plane, such that

a) $\mathbf{F} \neq \mathbf{0}$, $\operatorname{curl} \mathbf{F} = 0$ and $\operatorname{div} \mathbf{F} = 0$:

. ...

b) $\operatorname{curl} \mathbf{F} = 0$ and $\operatorname{div} \mathbf{F} > 0$:

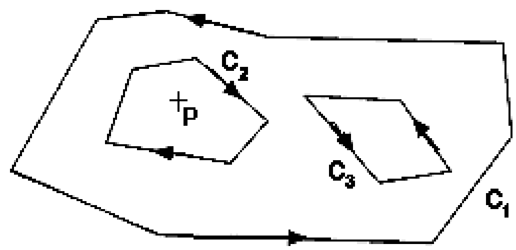
. ...

c) $\operatorname{curl} \mathbf{F} > 0$ and $\operatorname{div} \mathbf{F} = 0$:

. ...

5) Suppose that $M(x, y)$ and $N(x, y)$ are continuously differentiable functions, defined on the entire plane **except at a point** P **inside** C_2, such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$ (except at P). Orient the three closed curves C_1, C_2, C_3 as in the picture. Define $I_1 = \int_{C_1} M\,dx + N\,dy$, $I_2 = \int_{C_2} M\,dx + N\,dy$, $I_3 = \int_{C_3} M\,dx + N\,dy$ .What conclusion can you draw from the above information?

a) $I_1 < I_2 + I_3$
b) $I_2 < I_1 + I_3$
c) $I_3 < I_1 + I_2$
d) none of these.



6) The annual rate of fatal asthma attacks in England over the period 1862-1962 was 1 per 100,000 people. In 1963, five fatal attacks were observed in a study of 250,000 British people. Should 1963 be considered exceptional? Why? (For all options, X is the number of fatal asthma attacks observed among 250,000 people during a certain year.)

a) Yes; the probability of having exactly 5 fatal attacks in 1963 is very very small, definitely less than 5%.

b) Yes; the standard deviation of X is about 1.6 and $5 > 2(1.6)$ (using a 2σ test). Even if we use three standard deviations, we still have $5 > 3(1.6)$.

c) Borderline decision; the standard deviation of X is about 2.5; so 5 is just about "two sigmas". The answer becomes no if we use a 3σ test ($5 < 3(2.5)$).

d) No; the expected value of X is 2.5 and the standard deviation of X is about 1.6; the observed value 5 is inside the two sigma interval $[-0.7, 5.7]$ and well inside the three sigma interval $[-2.3, 7.3]$.

e) Definitely no; using a Poisson distribution, the expected value of X is 2.5 and the standard deviation of X is 2.5 as well; the observed value 5 is well inside both the two sigma interval $[0, 5]$ and the three sigma interval $[-2.5, 7.5]$.

7) Match each of the following objects to its description by filling in the appropriate letter (some of the descriptions may come up more than once, or not at all)

...... $x+y+z=5$	A circle
...... $x^2+y^2+z^2=5$	B straight line segment
...... $x^2-y^2-z^2=0$	C segment of a helix
...... $x=5\,,\ y=3\cos t\,,\ z=3\cos t\,;\ 0\le t\le\pi$	D planar surface
...... $x^2+y^2\le 5\,,\ -\infty<z<\infty$	E cylindrical surface
...... $\rho=5\,,\ 0\le\phi\le\pi/2\,,\ 0\le\theta<2\pi$	F solid cylinder
...... $\phi=\pi/4$	G conical surface
...... $\theta=\pi/4$	H sphere
...... $r=5\,,\ \theta=\pi/4\,,\ 0\le z\le 2$	I hemisphere
...... $\rho=5\,,\ \phi=\pi/4\,,\ 0\le\theta<2\pi$	
...... $r=5\,,\ 0\le\theta<2\pi\,,\ 1\le z\le 3$	
...... $x=5t\,,\ y=3\cos t\,,\ z=3\sin t\,;\ 0\le t\le\pi$	

3 Part II (152 points)

You should attempt all parts of all problems. Show your work!

[20 points] 1) In this problem, S denotes the level surface $f(x, y, z) = 1$ of the function $f(x, y, z) = x^2 + 3xy - 6z^2$, and P the point $(1, 2, 1)$. Note that P lies on the surface S.

a) Find the directional derivative of f in the direction of the vector $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ at P (Caution: $\mathbf{v}$ is not a unit vector; by definition, directional derivatives are calculated using unit vectors).

b) In which direction does f increase most rapidly at P?

c) Find a non-zero vector normal to S at P.

d) Find an equation describing the tangent plane to S at P.

e) Let $\mathbf{w}$ be a non-zero vector tangent to S at P. What can one say about the directional derivative of f in the direction of $\mathbf{w}$ at P?

[20 points] 2) Let D be the region described by the inequality

$$\frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{18} \leq 1 .$$

When the function $f(x, y, z) = \frac{x}{4} - \frac{y}{2} + \frac{z}{3} + 1$ is restricted to D, where does it assume its maximum and minimum values? What is the minimum value? the maximum value?

[16 points] 3) Let the continuous random variable X be the fraction of seats occupied in a Harvard examination room. (So X is between 0 and 1.) Suppose the probability density function for X is

$$f(x) = c\left(x - 2x^2 + x^3\right) \qquad \text{for } 0 \leq x \leq 1$$

where c is a constant.

a) Find the value of c.

b) Find the mean of X.

c) Find the variance of X.

d) What is the probability that the exam room is less than half full?

e) Is the median of X bigger or smaller than the mean?

[25 points] 4a) Is the vector field

$$\mathbf{F}(x, y, z) \;=\; 3\,x^2\,e^y\,\mathbf{i} \,+\, (x^3 + 1)e^y\,\mathbf{j} \,+\, z^2\,\mathbf{k} .$$

conservative? If yes, find a potential function, if no, give a reason.

4b) With $\mathbf{F}(x, y, z)$ as in part a), calculate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ along the curve

$$C \;:\; x = \cos t\,,\; y = \sin t\,,\; z = t\,, \quad 0 \leq t \leq 2\pi .$$

[25 points] 5) In this problem, $\mathbf{F}(x,y)$ denotes the vector field

$$\mathbf{F}(x,y) = \left(\frac{y-1}{x^2+(y-1)^2} - y \right) \mathbf{i} + \left(-\frac{x}{x^2+(y-1)^2} + x \right) \mathbf{j}\,.$$

a) What is the domain of $\mathbf{F}(x,y)$?
b) Compute the curl of $\mathbf{F}$ (hint: it is not identically equal to zero).
c) Compute the counterclockwise circulation of $\mathbf{F}(x,y)$ around the circle C_1 : $x^2+(y-1)^2=1$.
d) What is the counterclockwise circulation of $\mathbf{F}(x,y)$ around the circle C_2 : $x^2+y^2=81$?

[25 points] 6) Consider a test A intended to detect a certain disease with prevalence p $(0 < p < 1)$. Suppose that the specificity of A is higher than its sensitivity, say, $0 < a =$ (sensitivity of A) $< b =$ (specificity of A) < 1.

a) What is the predictive value positive of A?

b) Show that, if A is to be of any use, we should have $a+b \geq 1$. [Hint: compare the predictive value positive of A with the prevalence of the disease. Make sure all your steps are justified.]

If you are trying to design a test that is supposed to have a very high PV+, you might wonder if you should sacrifice sensitivity for specificity or vice-versa. In this context, the following two questions might be helpful.

c) For a fixed prevalence, show that the PV+ **increases** with the sensitivity a (for any fixed b) and **increases** with the specificity b (for any fixed a). To which one is the value of PV+ more "sensitive"?

d) If a second test B has sensitivity b and specificity a, can you compare the PV+ of B with the PV+ of A? How does the answer depend on the prevalence? Make sure you justify your answers.

Here is a table of the cumulative distribution function, $F(x)$, of the **standard** normal distribution $N(0,1)$ to be used in question 7.

x	$F(x)$	x	$F(x)$	x	$F(x)$
0.00	0.500	1.00	0.841	2.00	0.977
0.05	0.520	1.05	0.853	2.05	0.980
0.10	0.540	1.10	0.864	2.10	0.982
0.15	0.560	1.15	0.875	2.15	0.984
0.20	0.579	1.20	0.885	2.20	0.986
0.25	0.599	1.25	0.894	2.25	0.988
0.30	0.618	1.30	0.903	2.30	0.989
0.35	0.637	1.35	0.911	2.35	0.991
0.40	0.655	1.40	0.919	2.40	0.992
0.45	0.674	1.45	0.926	2.45	0.993
0.50	0.691	1.50	0.933	2.50	0.994
0.55	0.709	1.55	0.939	2.55	0.995
0.60	0.726	1.60	0.945	2.60	0.995
0.65	0.742	1.65	0.951	2.65	0.996
0.70	0.758	1.70	0.955	2.70	0.997
0.75	0.773	1.75	0.960	2.75	0.997
0.80	0.788	1.80	0.964	2.80	0.997
0.85	0.802	1.85	0.968	2.85	0.998
0.90	0.816	1.90	0.971	2.90	0.998
0.95	0.829	1.95	0.974	2.95	0.998

[21 points] 7) PKU is a genetic disease causing mental retardation in infants. Screening for PKU is done by measuring the amount of phenylalanine in the blood. For healthy children, the phenylalanine level is approximately normally distributed with mean 7 and standard deviation 3. For infants with PKU, the phenylalanine level is normal with mean 25 and standard deviation 10. Roughly 1 in 15,000 infants have PKU.

a) Suppose phenylalanine levels of 10 or greater are used to screen for PKU. What are the sensitivity and specificity of this screening test?

b) What are the predictive values of this screening test?

c) Suppose you are an obstetrician. What would you tell a mother of an infant who tests positive for PKU?