1 Math 21a, Fall 1998
Sample Final Exam

(May 17, 1996)

The exam has two parts. Part I consists of nine questions, some multiple choice, some questions with short answers. Problem 7 counts twelve points, the others six points each. Part II consists of six longer problems; problems 1 and 2 are worth 20 points each, and the other problems 25 points each. Please answer the questions in the spaces provided for this purpose. In part II, you must justify your answers.

Good luck!
Your name:

2 Part I (60 points)

Multiple choice/short questions – no justification is required. Each multiple choice question has only one correct answer.

1) $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are vectors in the plane as shown below. Note that $|\mathbf{A}| = |\mathbf{C}|$ and $|\mathbf{B}| = |\mathbf{D}|$.

In each case, circle the correct choice.

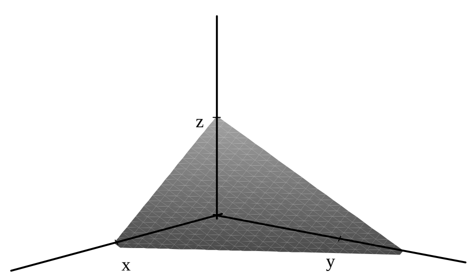
a) Which is greater?	$\mathbf{A}\cdot\mathbf{B}$	$\mathbf{C}\cdot\mathbf{D}$
b) Which is greater?	$\mathbf{A}\cdot\mathbf{B}$	$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B})$
c) Which has greater length?	$\mathbf{A}\times\mathbf{B}$	$\mathbf{C}\times\mathbf{D}$
d) Which has greater length?	$\mathbf{C}+\mathbf{D}$	$\mathbf{C}-\mathbf{D}$
e) Which is zero?	$(\mathbf{A}\times\mathbf{B})\times\mathbf{B}$	$\mathbf{A}\times(\mathbf{B}\times\mathbf{B})$
f) Which points up and out of the page?	$\mathbf{A}\times\mathbf{B}$	$\mathbf{D}\times\mathbf{C}$

2) All but one of the following parametrizations, with $-\infty < t < \infty$, trace out the same path in three dimensional Euclidean space. Circle the one that is different.

a) $x = t\,,\ y = t^2\,,\ z = t^3\,;$
b) $x = -t\,,\ y = -t^2\,,\ z = -t^3\,;$
c) $x = -t\,,\ y = t^2\,,\ z = -t^3\,;$
d) $x = t-1\,,\ y = (t-1)^2\,,\ z = (t-1)^3\,;$
e) $x = 2t\,,\ y = 4t^2\,,\ z = 8t^3\,.$

3) Only one of the following planes "cuts the corner off" the first octant. Which one?

a) $3x - y + 2z = 6\,;$
b) $3x + y + 2z = -6\,;$
c) $-3x - y + 2z = 6\,;$
d) $3x + y + 2z = 6\,;$
e) $3x + y - 2z = -6$



4) Give an example of a vector field $\mathbf{F} = \mathbf{F}(x, y)$ in the plane, such that

a) $\mathbf{F} \neq \mathbf{0}$, $\operatorname{curl} \mathbf{F} = 0$ and $\operatorname{div} \mathbf{F} = 0$:

. ...

b) $\operatorname{curl} \mathbf{F} = 0$ and $\operatorname{div} \mathbf{F} > 0$:

. ...

c) $\operatorname{curl} \mathbf{F} > 0$ and $\operatorname{div} \mathbf{F} = 0$:

. ...

5) Let $\mathbf{F}(x, y, z)$ be a vector field defined on all of three dimensional space, and $h(x, y, z)$ a scalar function, also defined on all of three dimensional space. Both $\mathbf{F}$ and h are at least twice continuously differentiable. Set $\mathbf{G} = \operatorname{curl} \mathbf{F}$ and $\mathbf{H} = \operatorname{grad} h$. Fill in the blanks with "grad", "curl", "div" to make each statement correct, or fill in "$\emptyset$" if the statement is correct as written.

a) ....$\mathbf{G} = 0$;

b) ....$\mathbf{H} = \mathbf{0}$;

c) ....$\mathbf{G} = \mathbf{0}$ if and only if $\mathbf{F}$ is conservative;

d) ....$h = \mathbf{0}$ at the critical points of h;

e) ....$\mathbf{F} = \mathbf{0}$ implies that $\mathbf{F} = \operatorname{grad} f$ for some scalar function f;

f) ....h is a conservative vector field.

6) In this problem S denotes the hemisphere $x^2 + y^2 + z^2 = 1$; $z \geq 0$ and D the flat disc $x^2 + y^2 \leq 1$; $z = 0$. Orient S so that $\mathbf{n}$ points away from the origin, and D with $\mathbf{n}$ pointing upward. Let $\mathbf{G}(x, y, z)$ be a continuously differentiable vector field, defined on all of Euclidean space, and $\mathbf{F} = \operatorname{curl} \mathbf{G}$. Which of the following is necessarily true?

a) $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_D \mathbf{F} \cdot \mathbf{n}\, d\sigma$;

b) $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = -\iint_D \mathbf{F} \cdot \mathbf{n}\, d\sigma$;

c) $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > \iint_D \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$;

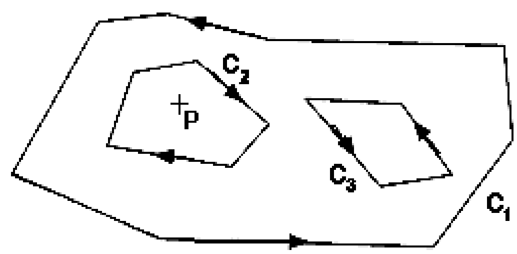
d) none of the above.

7) Match each of the following objects to its description by filling in the appropriate letter (some of the descriptions may come up more than once, or not at all)

...... $x+y+z=5$	A circle
...... $x^2+y^2+z^2=5$	B straight line segment
...... $x^2-y^2-z^2=0$	C segment of a helix
...... $x=5\,,\ y=3\cos t\,,\ z=3\cos t\,;\ 0\le t\le\pi$	D planar surface
...... $x^2+y^2\le 5\,,\ -\infty<z<\infty$	E cylindrical surface
...... $\rho=5\,,\ 0\le\phi\le\pi/2\,,\ 0\le\theta<2\pi$	F solid cylinder
...... $\phi=\pi/4$	G conical surface
...... $\theta=\pi/4$	H sphere
...... $r=5\,,\ \theta=\pi/4\,,\ 0\le z\le 2$	I hemisphere
...... $\rho=5\,,\ \phi=\pi/4\,,\ 0\le\theta<2\pi$	
...... $r=5\,,\ 0\le\theta<2\pi\,,\ 1\le z\le 3$	
...... $x=5t\,,\ y=3\cos t\,,\ z=3\sin t\,;\ 0\le t\le\pi$	

8) Suppose that $M(x,y)$ and $N(x,y)$ are continuously differentiable functions, defined on the entire plane **except at a point** P **inside** C_2, such that $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=1$ (except at P). Orient the three closed curves $C_1\,,\,C_2\,,\,C_3$ as in the picture. Define $I_1=\int_{C_1} M\,dx+N\,dy\,,\quad I_2=\int_{C_2} M\,dx+N\,dy\,,\quad I_3=\int_{C_3} M\,dx+N\,dy$ .What conclusion can you draw from the above information?

a) $I_1 < I_2 + I_3$
b) $I_2 < I_1 + I_3$
c) $I_3 < I_1 + I_2$
d) none of these.



9) Let S be the upper half (i.e., the portion lying above the $x-y$ plane) of the sphere of radius $\sqrt{2}$, centered at the origin. Orient S so that $\mathbf{n}$ points away from the origin. The flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ of the vector field

$$\mathbf{F}(x,y,z) \;=\; \frac{x\mathbf{i}\,+\,y\mathbf{j}\,+\,z\mathbf{k}}{x^2+y^2+z^2}$$

over S equals:
a) $\pi/2$;
b) $\pi/\sqrt{2}$;
c) π;
d) $\sqrt{2}\pi$;
e) 2π;
f) $2\sqrt{2}\pi$.

3 Part II (140 points)

Problems 1 and 2 are worth 20 points each, the others 25 points each. You should attempt all parts of all problems. Show your work!

1) In this problem, S denotes the level surface $f(x, y, z) = 1$ of the function $f(x, y, z) = x^2 + 3xy - 6z^2$, and P the point $(1, 2, 1)$. Note that P lies on the surface S.

a) Find the directional derivative of f in the direction of the vector $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ at P (Caution: $\mathbf{v}$ is not a unit vector; by definition, directional derivatives are calculated using unit vectors).

b) In which direction does f increase most rapidly at P?

c) Find a non-zero vector normal to S at P.

d) Find an equation describing the tangent plane to S at P

e) Let $\mathbf{w}$ be a non-zero vector tangent to S at P. What can one say about the directional derivative of f in the direction of $\mathbf{w}$ at P?

2) Let D be the the region described by the inequality

$$\frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{18} \leq 1 .$$

When the function $f(x, y, z) = \frac{x}{4} - \frac{y}{2} + \frac{z}{3} + 1$ is restricted to D, where does it assume its maximum and minimum values? What is the minimum value? the maximum value?

3a) Is the vector field

$$\mathbf{F}(x, y, z) = 3x^2 e^y \,\mathbf{i} + (x^3 + 1)e^y \,\mathbf{j} + z^2 \,\mathbf{k} .$$

conservative? If yes, find a potential function, if no, give a reason.

3b) With $\mathbf{F}(x, y, z)$ as in part a), calculate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ along the curve

$$C \,:\, x = \cos t ,\; y = \sin t ,\; z = t , \quad 0 \leq t \leq 2\pi .$$

4) Let S be the lower half of the sphere of radius 3 centered at the point $(0,0,3)$ – i.e., the surface described by the equation $z = 3 - \sqrt{9 - x^2 - y^2}$. Orient S so that the unit normal $\mathbf{n}$ points away from the center of the sphere. Evaluate the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ of the vector field

$$\mathbf{F}(x,y,z) \;=\; (y^2 + \sin(yz))\mathbf{i} \,+\, (x^2 + 3xz)\mathbf{j} \,+\, (2x + 4z)\mathbf{k}$$

over S (Hint: you probably would not want to do this by direct calculation).

5) In this problem, $\mathbf{F}(x,y)$ denotes the vector field

$$\mathbf{F}(x,y) \;=\; \left(\frac{y-1}{x^2+(y-1)^2} - y\right)\mathbf{i} \,+\, \left(-\frac{x}{x^2+(y-1)^2} + x\right)\mathbf{j}\,.$$

a) What is the domain of $\mathbf{F}(x,y)$?
b) Compute the curl of $\mathbf{F}$ (hint: it is not identically equal to zero).
c) Compute the counterclockwise circulation of $\mathbf{F}(x,y)$ around the circle C_1 : $x^2 + (y-1)^2 = 1$.
d) What is the counterclockwise circulation of $\mathbf{F}(x,y)$ around the circle C_2 : $x^2 + y^2 = 81$?

6) Let C denote the curve of intersection of the plane described by the equation $8x+4y+2z = 11$ and the cylinder $x^2+y^2 = 1$. Orient C counterclockwise when viewed from above. Compute the circulation of the vector field

$$\mathbf{F}(x,y,z) \;=\; (4xy + yz)\mathbf{i} \,+\, (yz - 6y^2 z)\mathbf{j} \,+\, (e^{\sin z} - 4x^3)\mathbf{k}$$

around C (Hint: doing this directly will be painful).