Math21a Sample Final 1 – Solutions

1 Multiple choice/short questions.

1) Let ϕ and ψ denote the angles between, respectively, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, $\mathbf{D}$. From the picture, $0 < \phi < \psi < \frac{\pi}{2}$, hence $0 < \sin\phi < \sin\psi$ and $0 < \cos\psi < \cos\phi$.

a) $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\phi = |\mathbf{C}||\mathbf{D}|\cos\phi > |\mathbf{C}||\mathbf{D}|\cos\psi = \mathbf{C}\cdot\mathbf{D}$ (because $|\mathbf{A}| = |\mathbf{C}|$, $|\mathbf{B}| = |\mathbf{D}|$ and $0 < \cos\psi < \cos\phi$). Correct answer: $\mathbf{A}\cdot\mathbf{B}$.

b) $\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\phi$ is strictly positive, but $\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0$ because $\mathbf{A}\perp(\mathbf{A}\times\mathbf{B})$. Correct answer: $\mathbf{A}\cdot\mathbf{B}$.

c) $|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}||\sin\phi| = |\mathbf{C}||\mathbf{D}||\sin\phi| < |\mathbf{C}||\mathbf{D}||\sin\psi| = |\mathbf{C}\times\mathbf{D}|$ (because $|\mathbf{A}| = |\mathbf{C}|$, $|\mathbf{B}| = |\mathbf{D}|$ and $0 < \sin\phi < \sin\psi$). Correct answer: $\mathbf{C}\times\mathbf{D}$.

d) The length of $\mathbf{C}+\mathbf{D}$ is the length of the longer of the two diagonals in the parallelogram spanned by $\mathbf{C}$ and $\mathbf{D}$, whereas the length of $\mathbf{C}-\mathbf{D}$ is the length of the shorter diagonal in the same parallelogram. One can also argue analytically: according to the law of cosines, $|\mathbf{C}+\mathbf{D}|^2 = |\mathbf{C}|^2 + |\mathbf{D}|^2 - 2|\mathbf{C}||\mathbf{D}|\cos(\pi-\psi)$ and $|\mathbf{C}-\mathbf{D}|^2 = |\mathbf{C}|^2 + |\mathbf{D}|^2 - 2|\mathbf{C}||\mathbf{D}|\cos\psi$. But $\cos\psi$ is strictly positive and $\cos(\pi-\psi)$ strictly negative, hence $|\mathbf{C}+\mathbf{D}|^2 > |\mathbf{C}|^2 + |\mathbf{D}|^2 > |\mathbf{C}-\mathbf{D}|^2$. Correct answer: $\mathbf{C}+\mathbf{D}$.

e) $\mathbf{B}\times\mathbf{B} = 0$, hence $\mathbf{A}\times(\mathbf{B}\times\mathbf{B}) = 0$. Note that $\mathbf{A}\times\mathbf{B}$ is non-zero and perpendicular to the plane of the page, hence not parallel to $\mathbf{B}$; this makes $(\mathbf{A}\times\mathbf{B})\times\mathbf{B}$ non-zero. Correct answer: $\mathbf{A}\times(\mathbf{B}\times\mathbf{B})$.

f) Right hand rule: curling the fingers of your right hand in the direction from $\mathbf{D}$ to $\mathbf{C}$ makes your right thumb stick up. Correct answer: $\mathbf{D}\times\mathbf{C}$.

2) Make the following substitutions for the parameter t in the parametrization a): $t\mapsto -t$ produces the parametrization c), $t\mapsto(t-1)$ the parametrization d), and $t\mapsto 2t$ the parametrization e). Only b) is different – correct answer: b).

3) The plane $3x+y+2z=6$ cuts the three coordinate axes in the points $(2,0,0)$, $(0,6,0)$ and $(0,0,3)$ – in each case, in the positive direction of the axis. Correct answer: d).

4)
a) Any constant vector field has zero curl and zero divergence, so $\mathbf{F} = \mathbf{i}$ is an example.

b) $\mathrm{curl}(M(x,y)\mathbf{i} + N(x,y)\mathbf{j}) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, and $\mathrm{div}(M(x,y)\mathbf{i} + N(x,y)\mathbf{j}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$, hence $M(x,y) = x$, $N(x,y) = 0$ gives the example $\mathbf{F}(x,y) = x\mathbf{i}$ of a vector field with zero curl and strictly positive divergence.

c) The vector field $\mathbf{F}(x,y) = x\mathbf{j}$ – i.e., with $M(x,y) = 0$, $N(x,y) = x$ has strictly positive curl but zero divergence.

5)
a) $\mathrm{div}\,\mathbf{G} = \mathrm{div}(\mathrm{curl}\,\mathbf{F}) = 0$, since $\mathrm{div}(\mathrm{curl}\,\mathbf{F}) = 0$ for any vector field $\mathbf{F}$.
b) $\mathrm{curl}\,\mathbf{H} = \mathrm{curl}(\mathrm{grad}\,h) = 0$, since $\mathrm{curl}(\mathrm{grad}\,h) = 0$ for any scalar function h.
c) Three dimensional space is simply connected, hence $\mathbf{F}$ is conservative if and only if $\mathbf{G} = \mathrm{curl}\,\mathbf{F} = 0$.
d) The critical points of the scalar function h are characterized by the vanishing of $\mathrm{grad}\,h$.
e) As in c), $\mathrm{curl}\,\mathbf{F} = 0$ precisely when $\mathbf{F}$ is conservative, i.e., when $\mathbf{F} = \mathrm{grad}\,f$ for some scalar function f.
f) By definition, the gradient of a scalar function is a conservative vector field.

6) The boundary ∂B of the half-ball B consists of the upper hemisphere S, oriented as specified, and the disk D, with the orientation opposite to the one specified in the problem – symbolically, $\partial B = S - D$. Note that $\mathrm{div}\,\mathbf{F} = \mathrm{div}(\mathrm{curl}\,\mathbf{G}) = 0$. The divergence theorem implies

$$0 = \iiint_B \mathrm{div}\,\mathbf{F} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma - \iint_D \mathbf{F}\cdot\mathbf{n}\,d\sigma\,,$$

so a) is the correct answer.

7)
a) The linear equation $x + y + z = 5$ describes a plane.
b) $x^2 + y^2 + z^2 = 5$ describes the sphere of radius $\sqrt{5}$, centered at the origin.
c) $x^2 - y^2 - z^2 = 0$ is equivalent to $x = \pm$distance from the x-axis, which describes a double cone.
d) $x = 5\,,\ y = 3\cos t\,,\ z = 3\cos t\,,\ 0 \le t \le \pi$ is a (slightly unusual) parametrization of a straight line segment.
e) $x^2 + y^2 \le 5$, $-\infty < z < \infty$, describes the solid cylinder of radius $\sqrt{5}$, centered on the z-axis.
f) $\rho = 5\,,\ 0 \le \phi \le \pi/2\,,\ 0 \le \theta < 2\pi$ describes the upper hemisphere of radius 5, centered at the origin.
g) $\phi = \pi/4$ describes a cone, centered on the z-axis.
h) $\theta = \pi/4$ describes a half-plane, bounded by the z-axis.

i) $r = 5\,,\ \theta = \pi/4\,,\ 0 \le z \le 2$, describes a straight line segment parallel to the z-axis.
j) $\rho = 5\,,\ \phi = \pi/4\,,\ 0 \le \theta < 2\pi$, describes a circle in a plane parallel to the x-y plane, centered at a point on the z-axis.
k) $r = 5\,,\ 0 \le \theta < 2\pi\,,\ 1 \le z \le 3$, describes a segment of the cylinder of radius 5, centered on the z-axis.
l) $x = 5t\,,\ y = 3\cos t\,,\ z = 3\sin t\,,\ 0 \le t \le \pi$, parametrizes one half of a winding of a helix.

8) Apply Green's theorem to R, the region between C_1 and C_2, and to R', the region enclosed by C_3. Since $\partial R = C_1 + C_2$ and $\partial R' = C_3$ (oriented as indicated),

$$\text{area}(R) = \iint_R dx\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{\partial R} M\,dx + N\,dy = I_1 + I_2\,,$$
$$\text{area}(R') = \iint_{R'} dx\,dy = \iint_{R'} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{\partial R'} M\,dx + N\,dy = I_3\,.$$

But the area of R is strictly greater than the area of R', so $I_1 + I_2 > I_3$, i.e., c) is the correct answer.

9) At every point (x, y, z), the vector field $\mathbf{F}(x, y, z)$ points away from the origin (except at the origin itself, where the vector field isn't defined). Thus, on S, $\mathbf{F}$ is a positive multiple of the outward unit normal $\mathbf{n}$. Note that

$$|\mathbf{F}(x, y, z)| = \frac{\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\,.$$

Hence, on S, $\mathbf{F}(x, y, z) = (\sqrt{2})^{-1}\mathbf{n}$ and $\mathbf{F} \cdot \mathbf{n} = (\sqrt{2})^{-1}$. Conclusion:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{1}{\sqrt{2}} \iint_S d\sigma = \frac{\text{area}(S)}{\sqrt{2}} = \frac{\text{area}\,\sqrt{2}}{2\sqrt{2}} = \frac{8\pi}{2\sqrt{2}}\,,$$

so f) is the correct answer.

2 Long questions

1)

a) The directional derivative of f in the direction of $\mathbf{v}$ is the dot product of grad f with the unit vector

$$\mathbf{u} \;=\; \frac{\mathbf{v}}{|\mathbf{v}|} \;=\; \frac{2\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{9}} \;=\; \frac{2}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{1}{3}\mathbf{k}\,.$$

On the other hand,

$$\operatorname{grad} f(1,2,1) \;=\; ((2x+3y)\mathbf{i}+3x\mathbf{j}-12z\mathbf{k})|_{(1,2,1)} \;=\; 8\mathbf{i}+3\mathbf{i}-12\mathbf{k}\,.$$

Taking the dot product, one finds that the directional derivative is

$$(8\mathbf{i}+3\mathbf{i}-12\mathbf{k})\cdot(\frac{2}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{1}{3}\mathbf{k}) \;=\; \frac{16}{3}-\frac{6}{3}-\frac{12}{3} \;=\; -\frac{2}{3}\,.$$

b) The function f increases most rapidly in the direction of grad $f(1,2,1)$, i.e., in the direction of the unit vector

$$\frac{\operatorname{grad} f(1,2,1)}{|\operatorname{grad} f(1,2,1)|} \;=\; \frac{8\mathbf{i}+3\mathbf{i}-12\mathbf{k}}{|8\mathbf{i}+3\mathbf{i}-12\mathbf{k}|} \;=\; \frac{8\mathbf{i}+3\mathbf{i}-12\mathbf{k}}{\sqrt{64+9+144}} \;=\; \frac{8}{\sqrt{217}}\mathbf{i}+\frac{3}{\sqrt{217}}\mathbf{j}-\frac{12}{\sqrt{217}}\mathbf{k}\,.$$

c) The gradient vector grad $f(1,2,1) = 8\mathbf{i}+3\mathbf{i}-12\mathbf{k}$ is non-zero and normal to S at $(1,2,1)$.

d) The tangent plane to S at $(1,2,1)$ is given by the linear equation

$$0 \;=\; ((x-1)\mathbf{i}+(y-2)\mathbf{j}+(z-1)\mathbf{k})\cdot\operatorname{grad} f(1,2,1) \;=\; 8(x-1)+3(y-2)-12(z-1)\,,$$

or equivalently, by $8x+3y-12z=2$.

e) Any tangent vector to S at $(1,2,1)$ is perpendicular to grad $f(1,2,1)$, making the directional derivative in any tangent direction equal to zero.

2) The gradient vector field $\operatorname{grad} f(x,y,z) = \frac{\mathbf{i}}{4} - \frac{\mathbf{j}}{2} + \frac{\mathbf{k}}{3}$ is everywhere non-zero. It follows that f has no critical points, and all maxima and minima of f on D must occur on the boundary. The boundary is given by $g(x,y,z) =_{\text{def}} \frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{18} = 1$. Apply Lagrange multipliers. We must check separately for points (x,y,z) on the boundary of D where $\operatorname{grad} g(x,y,z) = 0$ or $\operatorname{grad} g(x,y,z) = \lambda \operatorname{grad} f(x,y,z)$, for some constant λ. Since

$$\operatorname{grad} g(x,y,z) \;=\; \frac{x}{8}\mathbf{i} + \frac{y}{2}\mathbf{j} + \frac{z}{9}\mathbf{k}$$

vanishes only at the origin, there are no points of the first type. We need to solve the simultaneous equations $\operatorname{grad} g(x,y,z) = \lambda \operatorname{grad} f(x,y,z)$ and $g(x,y,z) = 1$ – in other words,

$$\frac{x}{8} = \frac{\lambda}{4}, \qquad \frac{y}{2} = -\frac{\lambda}{2}, \qquad \frac{z}{9} = \frac{\lambda}{3}, \qquad \frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{18} = 1\,.$$

Solving for x, y, z in terms of λ and substituting into the constraint gives $x = 2\lambda$, $y = -\lambda$, $z = 3\lambda$, and $1 = \lambda^2/4 + \lambda^2/4 + \lambda^2/2 = \lambda^2$, hence $\lambda = \pm 1$ and

$$(x,y,z) \;=\; (2,-1,3) \quad \text{or} \quad (x,y,z) \;=\; (-2,1\,-3)\,.$$

Since $f(2,-1,3) = 3$ and $f(-2,1,-3) = -3$, the function f assumes its maximum value at $(2,-1,3)$ and its minimum value at $(2,-1,3)$; these are the only extrema on D.

3)
a) The vector field $\mathbf{F}(x,y,z) = 3\,x^2\,e^y\,\mathbf{i} \;+\; (x^3+1)e^y\,\mathbf{j} \;+\; z^2\,\mathbf{k}$ has curl

$$\begin{aligned}
\operatorname{curl} \mathbf{F}(x,y,z) &= M(x,y,z)\mathbf{i} \;+\; N(x,y,z)\mathbf{j} \;+\; P(x,y,z)\mathbf{k}\,, \quad \text{with} \\
M(x,y,z) &= \left(\frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial z}((x^3+1)e^y)\right) \;=\; 0\,, \\
N(x,y,z) &= \left(\frac{\partial}{\partial z}(3x^2e^y) - \frac{\partial}{\partial x} z^2\right) \;=\; 0\,, \\
P(x,y,z) &= \left(\frac{\partial}{\partial x}((x^3+1)e^y) - \frac{\partial}{\partial y}(3x^2e^y)\right) \;=\; 0\,,
\end{aligned}$$

i.e., $\operatorname{curl} \mathbf{F}(x,y,z) = \mathbf{0}$. Since $\mathbf{F}$ is defined on all of Euclidean space, which is simply connected, this vector field is necessarily conservative. It is not difficult to find a potential function for $\mathbf{F}$. To do so, we need to solve the equations

$$\frac{\partial f}{\partial x} \;=\; 3x^2e^y\,, \qquad \frac{\partial f}{\partial y} \;=\; (x^3+1)e^y\,, \qquad \frac{\partial f}{\partial y} \;=\; z^2\,.$$

The most general solution of the first equation is $f(x,y,z) = x^3e^y + c(y,z)$. Substitution into the second equation gives $\frac{\partial}{\partial y}c(y,z) = e^y$, hence

$$f(x,y,z) \;=\; (x^3+1)e^y \;+\; d(z)$$

and finally $d'(z) = z^2$. Thus $f(x, y, y) = (x^3 + 1)e^y + \frac{z^3}{3}$ is a particular solution.

b) By the fundamental theorem of calculus, using the potential function found in part a), one finds

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \;=\; \int_C \operatorname{grad} f \cdot \mathbf{T}\, ds \;=\; f(\text{endpoint}) - f(\text{beginning point})$$
$$= \; f(1, 0, 2\pi) \;-\; f(1, 0, 0) \;=\; \frac{8}{3}\pi^3$$

4) Let R be the region above S and below the x-y plane, and D the disk of radius 3 in the plane $z = 3$, centered at $(0, 0, 3)$, oriented by the upward unit normal $\mathbf{k}$. Since $\partial R = S + D$, the divergence theorem implies

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \;+\; \iint_D \mathbf{F} \cdot \mathbf{n}\, d\sigma \;=\; \iiint_R \operatorname{div} \mathbf{F}\, dV\,.$$

It is easy to see that $\operatorname{div} \mathbf{F}(x, y, z) = 4$, hence

$$\iiint_R \operatorname{div} \mathbf{F}\, dV \;=\; 4\ \text{volume}(R) \;=\; 4\,\frac{1}{2}(\frac{4}{3}\,3^3\pi) \;=\; 72\,\pi\,.$$

On D, $\mathbf{n} = \mathbf{k}$ and $z = 3$, hence $\mathbf{F}(x, y, z) \cdot \mathbf{n} = 2x + 12$. By symmetry considerations (the function x is anti-symmetric about the plane $x = 0$!), then,

$$\iint_D \mathbf{F} \cdot \mathbf{n}\, d\sigma \;=\; \iint_{x^2+y^2 \le 9} (2x + 12)\, dx\, dy \;=\; \iint_{x^2+y^2 \le 9} 12\, dx\, dy \;=\; 12(9\,\pi)$$

Combining the three displayed equations, one finds

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \;=\; 72\,\pi \;-\; 108\,\pi \;=\; -36\,\pi\,.$$

5)

a) The vector field $\mathbf{F}(x,y)$ is defined on the entire plane except for the single point $(0,1)$, where the denominators vanish.

b)

$$\begin{aligned}\operatorname{curl}\mathbf{F}(x,y) &= \frac{\partial}{\partial x}\left(-\frac{x}{x^2+(y-1)^2}+x\right) - \frac{\partial}{\partial y}\left(\frac{y-1}{x^2+(y-1)^2}-y\right)\\ &= \frac{-(x^2+(y-1)^2)+2x^2}{(x^2+(y-1)^2)^2}+1+\frac{-(x^2+(y-1)^2)+2(y-1)^2}{(x^2+(y-1)^2)^2}+1 = 2\end{aligned}$$

c) Parametrize $C_1: \, x(t)=\cos t,\, y(t)=1+\sin t,\, 0\le t\le 2\pi$. Thus

$$\begin{aligned}\text{circulation} &= \int_{C_1}\left(\frac{y-1}{x^2+(y-1)^2}-y\right)dx + \left(\frac{-x}{x^2+(y-1)^2}+x\right)dy\\ &= \int_0^{2\pi}\left(\frac{\sin t}{\cos^2 t+\sin^2 t}-\sin t-1\right)(-\sin t)\,dt \quad +\\ &+ \int_0^{2\pi}\left(\frac{-\cos t}{\cos^2 t+\sin^2 t}+\cos t\right)(\cos t)\,dt\\ &= \int_0^{2\pi}\sin t\,dt = 0\,.\end{aligned}$$

d) We cannot use Green's theorem directly, since the vector field has a singularity in the interior of C_2. However, we can apply Green's theorem to the region R between the two circles. Since $\partial R = C_2 - C_1$,

$$\begin{aligned}\iint_R \operatorname{curl}\mathbf{F}\,dA &= \int_{C_2}\mathbf{F}\cdot\mathbf{T}\,ds - \int_{C_1}\mathbf{F}\cdot\mathbf{T}\,ds\\ &= \text{circulation around } C_2 - \text{circulation around } C_1\,.\end{aligned}$$

The circulation around C_1 vanishes according to c), so the circulation around C_2 equals

$$\iint_R \operatorname{curl}\mathbf{F}\,dA = 2\iint_R dA = 2(81\pi-\pi) = 160\pi\,.$$

6) The curve C is an ellipse running once around the cylinder, and lying entirely above the x-y plane. Let S denote the portion of the plane $8x + 4y + 2z = 11$ lying inside this ellipse, oriented by the normal vector which points (partially) upward. Then $\partial S = C$ (convince yourself that the orientations are consistent!). By Stokes' theorem,

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \;=\; \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma\,.$$

Let's calculate the curl:

$$\begin{aligned}
\operatorname{curl} \mathbf{F}(x,y,z) \;&=\; \left(\frac{\partial}{\partial y}(e^{\sin z} - 4x^3) - \frac{\partial}{\partial z}(yz - 6y^2 z)\right)\mathbf{i} \\
&+\; \left(\frac{\partial}{\partial z}(4xy + yz) \;-\; \frac{\partial}{\partial x}(e^{\sin z} - 4x^3)\right)\mathbf{j} \\
&+\; \left(\frac{\partial}{\partial x}(yz - 6y^2 z) \;-\; \frac{\partial}{\partial y}(4xy + yz)\right)\mathbf{k} \\
&=\; (6y^2 - y)\mathbf{i} \;+\; (y + 12x^2)\mathbf{j} \;-\; (4x + z)\mathbf{k}\,.
\end{aligned}$$

The surface S, is the portion of the graph $z = f(x,y) =_{\text{def}} \frac{11}{2} - 4x - 2y$ which lies above the disk $x^2 + y^2 \le 1$. Thus

$$\begin{aligned}
\operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma \;&=\; \operatorname{curl} \mathbf{F} \cdot (-f_x\, \mathbf{i} - f_y\, \mathbf{j} + \mathbf{k})\, dx\, dy \\
&=\; ((6y^2 - y)\mathbf{i} + (y + 12x^2)\mathbf{j} - (4x + z)\mathbf{k}) \cdot (4\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k})\, dx\, dy \\
&=\; (24y^2 - 4y + 2y + 24x^2 - 4x - z)\, dx\, dy \;=\; (24x^2 + 24y^2 - \frac{11}{2})\, dx\, dy
\end{aligned}$$

Use polar coordinates:

$$\begin{aligned}
\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma \;&=\; \iint_{x^2+y^2 \le 1} (24x^2 + 24y^2 - \frac{11}{2})\, dx\, dy \\
&=\; \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (24r^2 - \frac{11}{2}) r\, dr\, d\theta \;=\; 2\pi \left(6r^3 - \frac{11}{4} r^2\right)\Big|_0^1 \;=\; \frac{13}{2}\pi\,.
\end{aligned}$$

Conclusion:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \;=\; \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma \;=\; \frac{13}{2}\pi\,.$$