Solutions to Final Exam Review Problems
Spring 2001

(1) (a) Ricky's position at the given times are: $\mathbf{X}(\pi/4) = (\sqrt{2}/2, 1)$, $\mathbf{X}(\pi/2) = (0,0)$, $\mathbf{X}(\pi) = (-1,0)$, $\mathbf{X}(3\pi/2) = (0,0)$, and $\mathbf{X}(2\pi) = (1,0)$. (The path has period 2π and takes the shape of a figure-eight lying on its side, with crossing-point located at the origin.)

(b) $\mathbf{X}'(t) = -\sin t\mathbf{i} + 2\cos 2t\mathbf{j}$, so the speed $|\mathbf{X}'(t)| = \sqrt{\sin^2 t + 4\cos^2 2t}$. Thus $|\mathbf{X}'(\pi/2)| = |\mathbf{X}'(3\pi/2)| = \sqrt{5}$.

(c) $\mathbf{v_1} = \mathbf{X}'(\pi/2) = -\mathbf{i} - 2\mathbf{j}$ and $\mathbf{v_2} = \mathbf{X}'(3\pi/2) = \mathbf{i} - 2\mathbf{j}$, so $\cos\theta = (\mathbf{v_1}\cdot\mathbf{v_2})/(|\mathbf{v_1}||\mathbf{v_2}|) = 3/5$.

(d) $\int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 2t}\, dt$

(2) $\mathbf{v} = A\mathbf{k} \times (x\mathbf{i} + y\mathbf{j}) = -Ay\mathbf{i} + Ax\mathbf{j}$, so $\operatorname{div}\mathbf{v} = 0$ and $\operatorname{curl}\mathbf{v} = (0, 0, 2A)$.

(3) (a) The best answer is $\mathbf{X}(t) = (2t, 2/t)$ (for $1/2 \le t \le 2$).

(b) $\int_C \mathbf{F}\cdot d\mathbf{X} = \int_{1/2}^2 (8, 4t^2)\cdot(2, -2/t^2)\, dt = \int_{1/2}^2 16 - 8\, dt = 12$.

(c) $g(x,y) = x^2 y$ is such a function, so the integral in question is equal to $g(4,1) - g(1,4) = 12$.

(4) (Notice that f is undefined on the coordinate axes, so the minimum value of f occurs at a point where x and y are *strictly* positive.)

Check for critical points of f in the first quadrant: $\nabla f = \mathbf{0}$ implies $(-2/x^2 + y, -4/y^2 + x) = (0,0)$, in which case $x = 4/y^2 = 4/(4/x^4) = x^4$, to which $x = 1$ is the only possible solution in the first quadrant. This implies $y = 2$; the value of $f(1,2) = 6$. (One can verify that this is in fact a local minimum by way of the second derivative test, using the Hessian $H_f(x,y) = \begin{pmatrix} 4/x^3 & 1 \\ 1 & 8/y^3 \end{pmatrix}$.)

(5) The standard spherical-coordinate parameterization for the sphere is $\mathbf{X}(u,v) = (\cos v\sin u,\ \sin v\sin u,\ \cos u)$ for parameters $0 \le u \le \pi$ and $0 \le v \le 2\pi$, corresponding to ϕ and θ respectively. We compute

$$\begin{aligned}\mathbf{X}_u(u,v) &= (\cos v\cos u,\ \sin v\cos u,\ -\sin u),\\ \mathbf{X}_v(u,v) &= (-\sin v\sin u,\ \cos v\sin u,\ 0),\end{aligned}$$

so $(\mathbf{X}_u \times \mathbf{X}_v)(u,v) = \sin u\,(\cos v\sin u,\ \sin v\sin u,\ \cos v)$ and thus $dS = |\mathbf{X}_u \times \mathbf{X}_v|\, du\, dv = \sin u\, du\, dv$.

(a) The daytime half of the planet is the set of points where $0 \le u \le \pi$ and $0 \le v \le \pi$, and the area of this region is $\int_0^\pi \int_0^\pi \sin u\, du\, dv = 2\pi$. Thus, the average value of T is

$$\frac{1}{2\pi}\int_0^\pi \int_0^\pi (130\sin u + 15\sin u\sin v - 40)\sin u\, du\, dv = \frac{1}{2\pi}(65\pi^2 + 15\pi - 80\pi) = \frac{65(\pi - 1)}{2}.$$

(b) The *nighttime* portion of the ice cap is the set of points where $0 \le u \le \pi/6$ and $\pi \le v \le 2\pi$. The area of this region is $\int_\pi^{2\pi}\int_0^{\pi/6} \sin u\, du\, dv = \pi(1 - \frac{\sqrt{3}}{2})$. Thus, the average value of T is

$$\begin{aligned}&\frac{2}{\pi(2-\sqrt{3})}\int_\pi^{2\pi}\int_0^{\pi/6} (130\sin u + 15\sin u\sin v - 40)\sin u\, du\, dv\\ &= \frac{2}{\pi(2-\sqrt{3})}\left(130\pi\left(\frac{2\pi - 3\sqrt{3}}{24}\right) - 30\left(\frac{2\pi - 3\sqrt{3}}{24}\right) - 40\pi\left(1 - \frac{\sqrt{3}}{2}\right)\right)\\ &= \frac{130\pi^2 + 45\sqrt{3} - 510\pi + 45\pi\sqrt{3}}{6\pi(2-\sqrt{3})}.\end{aligned}$$

(Now you know why this problem didn't make the cut!)

(6) We use the method of Lagrange multipliers to build a list of candidates for the extreme points of T. First we calculate $\nabla T = (4x,\ z,\ y-4)$. Next, the constraint equation with denominators cleared is $4x^2+y^2+4z^2-16=0$, and the gradient of the constraint is $(8x,\ 2y,\ 8z)$. Thus, we must solve the system

$$\begin{aligned} 4x &= \lambda \cdot 8x, \\ z &= \lambda \cdot 2y, \\ y-4 &= \lambda \cdot 8z, \\ 4x^2+y^2+4z^2 &= 16. \end{aligned}$$

From the first equation, we find that $x=0$ or $\lambda = 1/2$. In the latter case, we find that $y=z$ and $y-4=4z$, so $y=z=-4/3$ and hence $4x^2 = 16-5(-4/3)^2 = 64/9$, so we place the points $(4/3,-4/3,-4/3)$ and $(-4/3,-4/3,-4/3)$ on our list of candidates.

If instead $x=0$, we can combine the second and third equations to find that $4z^2 = 8\lambda yz = (y-4)y$, and combining this with the constraint equation gives $16-y^2 = 4z^2 = (y-4)y$, which implies that $y=4$ or $y=-2$. Thus, we can find the corresponding values of z and place the points $(0,4,0)$, $(0,-2,\sqrt{3})$ and $(0,-2,-\sqrt{3})$ on our list of candidates.

By testing these five candidates, we now see that the coldest spot is located at $(0,-2,\sqrt{3})$, where $T = 150-6\sqrt{3}$, and the hottest spots are at $(\pm 4/3,-4/3,-4/3)$, where $T = 160\frac{2}{3}$.

(7) (You may assume that the problem intends that R lie entirely within the first quadrant (as opposed to also containing points that lie in the third quadrant), so that there is a domain in the first quadrant of the uv-plane that corresponds to the region R.)

Any point (x,y) in R satisfies $1 \le xy \le 9$ and $1 \le y/x \le 4$. In terms of u and v, we find that this is equivalent to saying that $1 \le u^2 \le 9$ and $1 \le v^2 \le 4$, or, since u and v can be taken to be positive, $1 \le u \le 3$ and $1 \le v \le 2$. This is the uv-domain that corresponds to R.

The Jacobian matrix of the coordinate transformation is $\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 1/v & -u/v^2 \\ v & u \end{pmatrix}$, which has determinant $2u/v$. Thus, in terms of u and v, the given integral is

$$\int_1^3 \int_1^2 (v+u) \cdot 2u/v \, dv \, du = \int_1^3 2u + 2u^2 \ln 2 \, du = 8 + \frac{52 \ln 2}{3}.$$

(8) The square γ has vertices $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$, and bounds a region R of area $(\sqrt{2})^2 = 2$. The field $\mathbf{F}$ has constant circulation density equal to 2, so by Green's theorem, $\int_\gamma \mathbf{F} \cdot d\mathbf{X} = \iint_R 2\, dA = 2Area(R) = 4$.

(9) (a) $\iint_R f\, dA = \int_{-1}^1 \int_0^{1-x^2} y\, dy\, dx = (1/2)\int_{-1}^1 (1-x^2)^2\, dx = 8/15$.

(b) The triangle in the uv-plane has vertices $A(-1,1)$, $B(0,0)$, and $C(1,1)$, which correspond to the xy-points $(-1,0)$, $(0,0)$, and $(1,0)$ respectively. The triangle edge joining A and C corresponds to the parabola portion of the boundary of R; the edge joining A and B (resp. B and C) corresponds to the negative (resp. positive) x-axis portion of the boundary of R.

(c) The Jacobian is

$$\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} v & u \\ -2u & 2v \end{pmatrix},$$

and its determinant is $2v^2 - (-2u^2) = 2(u^2+v^2)$.

(d) The integral is

$$\int_0^1 \int_{-v}^{v} (v^2-u^2)(2u^2+2v^2)\, du\, dv = 2\int_0^1 \int_{-v}^{v} v^4 - u^4\, du\, dv = \frac{16}{5}\int_0^1 v^5\, dv = \frac{8}{15}.$$

(10) (a) Average value of $x^2+y^2 = \frac{\iint_S (x^2+y^2)\, dS}{\iint_S dS}$.

(b) The parameterization is $\mathbf{X}(\phi,\theta) = (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi)$ for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. We compute

$$\mathbf{X}_\phi(\phi,\theta) = (a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -a\sin\phi),$$
$$\mathbf{X}_\theta(\phi,\theta) = (-a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0),$$

so $(\mathbf{X}_\phi \times \mathbf{X}_\theta)(\phi,\theta) = a^2\sin\phi\,(\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$ and thus $dS = |\mathbf{X}_\phi \times \mathbf{X}_\theta|\,d\phi\,d\theta = a^2\sin\phi\,d\phi\,d\theta$.

(c) The denominator $\iint_S dS$ is just the surface area of the sphere of radius a, which is $4\pi a^2$, so the average value is

$$\frac{1}{4\pi a^2}\int_0^{2\pi}\int_0^{\pi} a^2(\cos^2\theta+\sin^2\theta)\sin^2\phi\cdot a^2\sin\phi\,d\phi\,d\theta = \frac{a^2}{2}\int_0^{\pi}\sin^3\phi\,d\phi$$
$$= \frac{a^2}{2}\int_0^{\pi}\sin\phi - \sin\phi\cos^2\phi\,d\phi = \frac{a^2}{2}\left(2-\frac{2}{3}\right) = \frac{2a^2}{3}.$$

(11) It is important to note that in the formulation of the pressure $P(z)$, increasing depth corresponds to increasing *positive* z. Thus, we imagine coordinate axes positioned so that the water's surface is the plane $z = 0$ and the positive z-axis points into the sea. (Thus, a negative z-component of force on a submerged object means it is a force directed *towards* the surface, as we will calculate.) Let S denote the surface of the submerged object, and let R denote the solid region it occupies.

(a) $\mathbf{F} \approx \sum(-P(z)\,\Delta S)\,\mathbf{n} = \sum(-P(z)\mathbf{n})\,\Delta S$

(b) The component F_z can be written as the flux of the vector field $-P(z)\mathbf{k}$ *out* of S:

$$F_z = \mathbf{F}\cdot\mathbf{k} = \left(\iint_S -P(z)\mathbf{n}\,dS\right)\cdot\mathbf{k} = \iint_S (-P(z)\mathbf{k})\cdot\mathbf{n}\,dS.$$

(c) By the divergence theorem,

$$F_z = \iint_S (-P(z)\mathbf{k})\cdot\mathbf{n}\,dS = \iiint_R \operatorname{div}(-P(z)\mathbf{k})\,dV$$
$$= \iiint_R \frac{\partial}{\partial z}(-P_0 - wz)\,dV$$
$$= \iiint_R -w\,dV = -w\,Vol(R).$$

(d) $F_x = F_y = 0$. For example,

$$F_x = \mathbf{F}\cdot\mathbf{i} = \left(\iint_S -P(z)\mathbf{n}\,dS\right)\cdot\mathbf{i}$$
$$= \iint_S (-P(z)\mathbf{i})\cdot\mathbf{n}\,dS$$
$$= \iiint_R \operatorname{div}(-P(z)\mathbf{i})\,dV$$
$$= \iiint_R \frac{\partial}{\partial x}(-P_0 - wz)\,dV = \iiint_R 0\,dV = 0.$$

(e) The weight force is clearly directed along the positive z-axis; since $F_x = F_y = 0$ and F_z is negative, the pressure force is indeed opposite to the weight. The magnitude of F is equal to the magnitude of F_z, which is equal to the weight of water displaced by the object, as predicted by Archimedes.

(12) (a) We parameterize the boundary of R by the three pieces

$$\mathbf{X_1}(t) = t\mathbf{i},\ 0 \le t \le 1,$$
$$\mathbf{X_2}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \pi/2,$$
$$\mathbf{X_3}(t) = (1-t)\mathbf{j},\ 0 \le t \le 1.$$

Using these parameterizations, we compute the total line integral to be

$$\int_0^1 \mathbf{F}(t,0)\cdot(1,0)\,dt + \int_0^{\pi/2} \mathbf{F}(\cos t, \sin t)\cdot(-\sin t, \cos t)\,dt + \int_0^1 \mathbf{F}(0,1-t)\cdot(0,-1)\,dt$$

$$= \int_0^1 t\,dt + \int_0^{\pi/2} -\cos^3 t \sin t\,dt + \int_0^1 0\,dt = 1/2 - 1/4 = 1/4.$$

(b) The circulation density of $\mathbf{F}$ equals $0 - \frac{\partial}{\partial y}(x(1-y^2)) = 2xy$, and so by Green's theorem, the line integral in question is equal to

$$\iint_R 2xy\,dA = \int_0^{\pi/2}\int_0^1 2r^2\cos\theta\,\sin\theta\,r\,dr\,d\theta = \int_0^1 r^3\,dr \int_0^{\pi/2} \sin 2\theta\,d\theta = 1/4.$$

(13) We evaluate the innermost integral to obtain

$$\frac{e^{2\ln 3}-1}{2}\int_0^1\int_{\sqrt[3]{z}}^1 \frac{\pi\sin(\pi y^2)}{y^2}\,dy\,dz.$$

To evaluate this double integral, we first observe that the region R in the yz-plane over which it is delimited is $\{0 \le z \le 1,\ \sqrt[3]{z} \le y \le 1\}$, i.e., the region bounded by the curve $z = y^3$, the y-axis, and the line $y = 1$. Thus we can reverse the order of integration by rewriting the bounds for R as $\{0 \le y \le 1,\ 0 \le z \le y^3\}$; our integral becomes

$$\frac{9-1}{2}\int_0^1\int_0^{y^3} \frac{\pi\sin(\pi y^3)}{y^2}\,dz\,dy = 4\pi\int_0^1 y\sin(\pi y^2)\,dy = 4.$$