

## Solutions to Math 21a Exam #1 -- Spring 2001

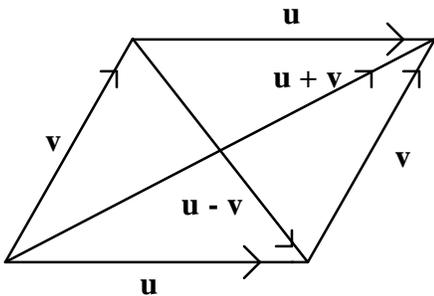
- (1) (16 pts) Given the following points:  $P(1, 2, -2)$ ,  $Q(-3, 1, -14)$ ,  $R(-1, 2, -6)$ , and  $S(1, -2, -8)$ . Are these points coplanar, i.e. do they all lie in a common plane in  $\mathbf{R}^3$ ? Justify your answer.

**Solution:** There were many ways to approach this problem. For example, you could use three of the four points to find two difference vectors, take their cross product to get a normal vector. Then use that vector and one of the points to get an equation for the plane containing the three points. Then plug in the coordinates of the fourth point to see whether it's in the plane. Specifically,  $\mathbf{PQ} = (-4, -1, -12)$  and  $\mathbf{PR} = (-2, 0, -4)$ . Their cross product gives the vector  $(4, 8, -2)$ , so take the normal vector to be half of that, i.e.  $\mathbf{n} = (2, 4, -1)$ . Using the point  $P$ , we get the equation for the plane  $2x + 4y - z = 12$ . It's then easy to see that the point  $S$  does not lie on this plane, so the points are not coplanar.

Another approach would be to consider the three vectors  $\mathbf{PQ}$ ,  $\mathbf{PR}$ , and  $\mathbf{PS}$ . The volume of the parallelepiped determined by these three vectors will be nonzero if and only if the four points are not coplanar. Using the triple scalar product to find the volume of the parallelepiped, we find that the volume is 20, hence the points are not coplanar.

- (2) (16 pts) Use vectors to show that the diagonals of a parallelogram are perpendicular if and only if all four sides of the parallelogram have equal length. (Such a figure is called a rhombus.)

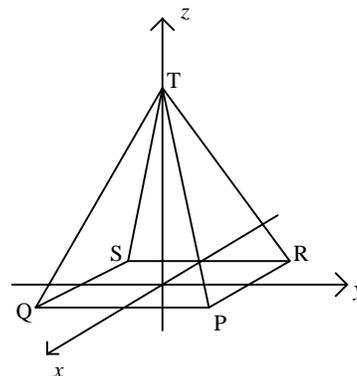
**Solution:**



If two sides are determined by vectors  $\mathbf{u}$  and  $\mathbf{v}$ , then the two diagonal vectors of the parallelogram will be  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$ . Thus, the two diagonals are perpendicular if and only if  $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$ . By multiplying this out we see that this is the case if and only if  $\mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = 0$ . But this says that  $\|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$ , or simply that  $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$ . But this will be the case if and only if  $\|\mathbf{u}\| = \|\mathbf{v}\|$ . Therefore, the diagonals will be perpendicular if and only if these two sides, and hence all four sides, are of equal length.

- (3) (16 pts) Consider a pyramid-shaped tent of height 5 meters and with a 4 meter by 4 meter square base.
- (a) The top of the tent lies directly over the center of the base. Find the angle between two adjacent triangular faces of the tent, i.e. the angle between their outward pointing normal vectors.

**Solution:** There are many acceptable ways to set up this problem, but most involve situating the four corners of the base of the tent and the top point at specific coordinates in  $\mathbf{R}^3$ . For example, we could locate the four corners of the base at  $P(2, 2, 0)$ ,  $Q(2, -2, 0)$ ,  $R(-2, 2, 0)$ , and  $S(-2, -2, 0)$  and the top point at  $T(0, 0, 5)$ . With these choices, we can find an outward normal  $\mathbf{n}_1 = \mathbf{QP} \times \mathbf{QT}$  for the front face and an outward normal  $\mathbf{n}_2 = \mathbf{PR} \times \mathbf{PT}$  for the right face. Simple calculations give that  $\mathbf{n}_1 = (20, 0, 8)$  and  $\mathbf{n}_2 = (0, 20, 8)$ . To find the angle  $\theta$  between these two vectors, we calculate  $\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{64}{464} = \frac{4}{29}$ , so we get that  $\theta = \cos^{-1}(\frac{4}{29})$



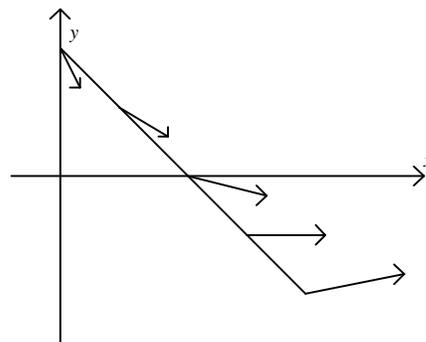
- (b) If the tent is made of canvas and consists of the four triangular sides plus the square bottom, find the total area of canvas used to make the tent.

**Solution:** The area of each triangular face is one-half the area of the parallelogram determined by a pair of edge vectors. For example, for the front face we calculate an area of  $\frac{1}{2} \|\mathbf{QP} \times \mathbf{QT}\| = 2\sqrt{29}$ . Thus when we include the area of four identical triangular sides and the  $4 \times 4$  square base, we get a total of  $16 + 8\sqrt{29}$  square meters of canvas.

- (4) (16 pts) Consider the vector field  $\mathbf{F}(x, y) = (y + 2x)\mathbf{i} + (x - 3)\mathbf{j}$ . We wish to find the work integral  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{x}$  along the path  $\gamma$ , where  $\gamma$  is the line segment from the point  $(0, 2)$  to the point  $(4, -2)$ .

- (a) Sketch the direction of  $\mathbf{F}$  at the five points  $(0, 2)$ ,  $(1, 1)$ ,  $(2, 0)$ ,  $(3, -1)$ , and  $(4, -2)$ . On the basis of this sketch, explain whether you expect the value of the integral to be positive, negative, or zero.

**Solution:** Since the work integral  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{x}$  is derived from  $\int_{\gamma} \mathbf{F}_T ds$  or, equivalently  $\int_a^b \mathbf{F} \cdot \mathbf{v} dt$  where  $\mathbf{v}$  is the velocity vector for a parametrization of the given path  $\gamma$ , we can see from the diagram that the integrand will always be positive and, hence, the work integral will be positive.



- (b) Evaluate the integral  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{x}$  along the line segment from the point  $(0, 2)$  to  $(4, -2)$ .

**Solution:** The simplest parametrization of the line segment is  $\begin{cases} x = 4t \\ y = 2 - 4t \end{cases}$  where  $t \in [0, 1]$ .

We calculate  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{x} = \int_{\gamma} (y + 2x) dx + (x - 3) dy = \int_0^1 [(2 + 4t)(4) + (4t - 3)(-4)] dt = \int_0^1 20 dt = 20$ .

(5) (18 pts) Consider the function  $f(x, y) = \sqrt{50 - x^2 - y^2}$ .

(a) Describe geometrically the surface formed by the graph  $z = f(x, y)$ .

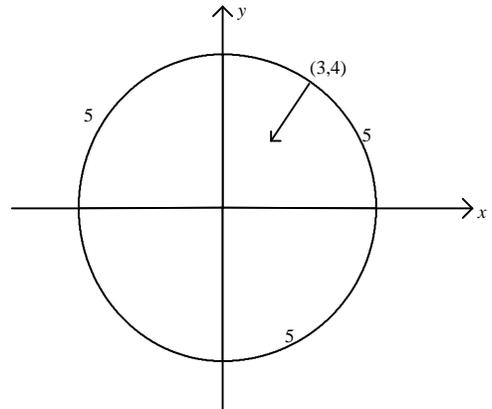
**Solution:** If we square both sides of the graph equation  $z = \sqrt{50 - x^2 - y^2}$ , we get the upper half of the sphere whose equation is  $x^2 + y^2 + z^2 = 50$ . This center of this sphere is at the origin and its radius is  $5\sqrt{2}$ .

(b) Sketch and label the contour of the function  $f$  through the point  $(x, y) = (3, 4)$ .

(c) Calculate  $\nabla f$ , the gradient of  $f$ , at the point  $(x, y) = (3, 4)$  and indicate it on your diagram above.

**Solution:**  $\nabla f = \left( \frac{-x}{\sqrt{50 - x^2 - y^2}}, \frac{-y}{\sqrt{50 - x^2 - y^2}} \right)$ , so at  $(3, 4)$

we have  $\nabla f(3, 4) = \left(-\frac{3}{5}, -\frac{4}{5}\right)$



(d) Calculate the directional derivative of  $f$  at the point  $(3, 4)$  in the direction of the vector  $(2, -1)$ .

**Solution:**  $(Df)_{\mathbf{u}}(3, 4) = \nabla f(3, 4) \cdot \mathbf{u}$  where  $\mathbf{u}$  is a unit vector in the direction of the vector  $(2, -1)$ , i.e.

$\mathbf{u} = \frac{(2, -1)}{\sqrt{5}}$ . Putting this together with the result above, we compute  $(Df)_{\mathbf{u}}(3, 4) = -\frac{2}{5\sqrt{5}}$

(e) Find an equation for the plane tangent to the graph  $z = f(x, y)$  at the point on the graph where  $x = 3$  and  $y = 4$ .

**Solution:** There are several ways to do this. From the expression for linearization, we can use the partial derivatives given above to get that the tangent plane has equation  $z = 5 - \frac{3}{5}(x - 3) - \frac{4}{5}(y - 4)$ .

Alternatively, we can take the equation for the sphere  $x^2 + y^2 + z^2 = 50$  and treat this as a level surface for a function of three variables. Using the gradient of this new function and the point  $(3, 4, 5)$  on the graph surface, we can get the normal vector  $\mathbf{n} = (6, 8, 10)$  or, more simply  $\mathbf{n} = (3, 4, 5)$ . Using this normal vector and the point  $(3, 4, 5)$ , we get the equation  $3(x - 3) + 4(y - 4) + 5(z - 5) = 0$ . A little algebra shows that this agrees with the equation of the tangent plane we obtained above.

(6) (18 pts) The Texahoma Saloon is situated on the Texas-Oklahoma border. Its roof is perpendicular to the vector  $(1, -2, 2)$  and includes the point with coordinates  $(-2, 4, 5)$ . [The  $x$ -axis runs from west to east, the  $y$ -axis from south to north, and the  $z$ -axis points vertically upward.] Marshall Wyatt Zorn has been sent from Minnesota to this border country to enforce a recent executive order decreeing that all distances are to be measured in meters.

(a) Write an equation satisfied by the coordinates  $(x, y, z)$  of any point on the roof.

**Solution:** Using  $\mathbf{n} = (1, -2, 2)$  and the point  $(-2, 4, 5)$ , we get the equation  $x - 2y + 2z = 0$  for the roof.

(b) The state line lies along the plane  $y = 4$  (with  $y > 4$  in Oklahoma and  $y < 4$  in Texas, of course). Marshall Zorn is asked to paint the state line on the saloon's roof. Find parametric equations for this line. (You do not have to specify a parameter range.)

**Solution:** We have the equations  $\begin{cases} x - 2y + 2z = 0 \\ y = 4 \end{cases}$  which can be turned into parametric equations by letting  $z = t$  and solving for the rest. That is,  $\begin{cases} x = 8 - 2t \\ y = 4 \\ z = t \end{cases}$ .

(c) Prominently posted in the saloon is a sign that states "No firearms within 2 meters of the roof." Calamity W. Jane, while trying to make a point about the Second Amendment, holds her Colt 45 at the point  $(-3, -3, 2)$ . Marshall Zorn needs to decide whether she is in violation of the law. Help him out by calculating the distance from the Colt to the roof.

**Solution:** The easiest way to do this is to choose any point on the roof, such as  $(6, 4, 1)$ , find the difference vector from  $(-3, -3, 2)$  to this point, namely  $(9, 7, -1)$ , and then calculate the scalar projection of this vector in the direction perpendicular to the planar roof. This gives us

$$(9, 7, -1) \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} = (9, 7, -1) \cdot \frac{(1, -2, 2)}{3} = \frac{9 - 14 - 2}{3} = -\frac{7}{3},$$

so the distance must be  $\frac{7}{3}$  which is greater than 2, so Calamity is within the law.

(d) In panic at the prospect of being arrested, Jane fires her Colt 45. The bullet travels with great velocity proportional to the vector  $(5, 6, 2)$  and moves in a straight line through the roof. Determine the location of the hole that it makes.

**Solution:** We can parametrize the line using the point  $(-3, -3, 2)$  and the vector  $(5, 6, 2)$  to get the

parametric equations  $\begin{cases} x = -3 + 5t \\ y = -3 + 6t \\ z = 2 + 2t \end{cases}$ . Substituting these into the equation for the roof gives

$$(-3 + 5t) - 2(-3 + 6t) + 2(2 + 2t) = 0, \text{ which we can solve to get } t = \frac{7}{3}.$$

This yields the point on the roof  $(x, y, z) = (\frac{26}{3}, 11, \frac{20}{3})$ .