Solutions to Math 21a Exam #2 - Spring 2001

(1) (16 pts) The President has opened up a wildlife preserve in Alaska for oil drilling. In retaliation, Democrats introduce a bill to allow oil drilling on the Ellipse, a park between the White House and the Washington Monument bounded by the curve $x^2 + 4y^2 = 100$. Each political party is to be given one drilling site.

The President consults geologists from Texas, who inform him that the value V of the oil from a well drilled on the Ellipse will be given by the formula $V = 200 + 18y - x^2 - y^2$.

(a) He thereupon signs the bill into law and instructs you, his Secretary of Energy, to find the coordinates x and y for the most valuable drilling site(s) (for the Republicans) and of the least valuable site(s) (for the Democrats) and to tell him the maximum and minimum values. Do so.

Solution: The function to be optimized is $V(x, y) = 200 + 18y - x^2 - y^2$. You must look for stationary points in the interior of the region and then look on the boundary of the given region. For stationary points, we calculate:

$$\frac{\partial V}{\partial x} = -2x \quad \text{and} \quad \frac{\partial V}{\partial y} = 18 - 2y\,,$$

so the only point where they both vanish is at $(x, y) = (0, 9)$. However, this point is outside the elliptical region.

Next we use the Method of Lagrange Multipliers to find candidates for maxima and minima on the boundary of the region. Letting $g(x, y) = x^2 + 4y^2$, we must satisfy the relations $\nabla V = \lambda \nabla g$ and $g(x, y) = 100$. This gives us:

$$-2x = \lambda(2x)\,, \quad 18 - 2y = \lambda(8y)\,, \quad \text{and} \quad x^2 + 4y^2 = 100.$$

The first of these gives us $2x(\lambda + 1) = 0$, so either $x = 0$ or $\lambda = -1$. In the first case, plugging $x = 0$ into the constraint gives us that either $y = 5$ or $y = -5$, so we have the candidates (0, 5) and (0,-5). In the second case, if $\lambda = -1$, then we must have $18 - 2y = -8y$, so $y = -3$. Plugging this into the constraint, we get that either $x = -8$ or $x = 8$, so we have the candidates (-8,-3) and (8,-3).

Finally, we compare the values: $V(0, 5) = 265$, $V(0,-5) = 85$, $V(-8,-3) = 73$, and $V(8,-3) = 73$. So the Republicans will want to drill at the point (0,5) to get the maximum value and they will relegate the Democrats to drill either at the point (-8,-3) or (8,-3).

(b) The President also wants to know whether the Republicans could do even better if allowed to drill outside the Ellipse? Determine if there is a drilling site with even greater V, and if so, where it is located.

Solution: If allowed to drill outside The Ellipse, the Republicans will surely want to drill at the point (0,9) since this will give the value $V(0, 5) = 281$. A quick check of the Hessian matrix will convince you that this is a local maximum and a closer look at the quadratic nature of the function should convince you that this must also be the global maximum.

(2) (14 pts) The function $F(x,y)=x^2y-4xy+3x^2+\frac{1}{2}y^2$ has three stationary points, at x = 0, 1, and 5.

(a) Find the values of y at these three stationary points.

(b) Classify each stationary point as a maximum, minimum, or saddle point.

Solution: The stationary points will be given by the simultaneous solutions of the equations:

$$\left\{\begin{matrix} F_x = 2xy-4y+6x=0 \\ F_y = x^2-4x+y=0 \end{matrix}\right\}.$$

Normally these would be somewhat difficult to solve, but we are given that the solutions will occur where x = 0, 1, and 5. Using the second equation, for instance, we'll get the corresponding y-values of 0, 3, and -5. (You might also check that the first equation is consistent.) We thus have the three stationary point (0, 0), (1, 3), and (5,-5).

Next, we calculate the Hessian matrix of 2nd partial derivatives at each of these points:

$$H_F(x,y)=\begin{bmatrix} 2y+6 & 2x-4 \\ 2x-4 & 1 \end{bmatrix}$$

Evaluating this at each stationary point, we have:

$H_F(0,0)=\begin{bmatrix} 6 & -4 \\ -4 & 1 \end{bmatrix}$ with determinant = -10 < 0, so this gives a <u>saddle point</u>.

$H_F(1,3)=\begin{bmatrix} 12 & -2 \\ -2 & 1 \end{bmatrix}$ with determinant = 8 > 0, and $F_{xx}>0$, so this gives a <u>local minimum</u>.

$H_F(5,-5)=\begin{bmatrix} -4 & 6 \\ 6 & 1 \end{bmatrix}$ with determinant = -40 < 0, so this also gives a <u>saddle point</u>.

(3) (14 pts) A function $F(x, y)$ is given by the formula $F(x, y) = g(x^2 + y^2)$, where g is a twice-differentiable function of one variable. Express $\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}$ in terms of x, y, and the first and second derivatives of g.

Solution: The most important step in solving this problem is to understand what the variables and dependencies are. To do this, we can start by giving a name to the argument of the function g. That is, let $u = x^2 + y^2$. Then we have the following schematic:

$$(x,y)\to u(x,y)\to g(u(x,y))=F(x,y)$$

We therefore must use the Chain Rule to calculate the derivatives of F with respect to the variables x and y. We calculate:

$$\frac{\partial F}{\partial x}=\frac{dg}{du}\frac{\partial u}{\partial x}=2x\frac{dg}{du} \quad \text{and} \quad \frac{\partial F}{\partial y}=\frac{dg}{du}\frac{\partial u}{\partial y}=2y\frac{dg}{du}.$$

Taking the derivatives again (and again applying the Chain Rule appropriately), we calculate:

$$\frac{\partial^2 F}{\partial x^2}=\frac{\partial}{\partial x}\left(2x\frac{dg}{du}\right)=2\frac{dg}{du}+2x\frac{\partial}{\partial x}\left(\frac{dg}{du}\right)=2\frac{dg}{du}+2x\left(\frac{d^2g}{du^2}\frac{\partial u}{\partial x}\right)=2g'(u)+4x^2g''(u)$$

$$\frac{\partial^2 F}{\partial y^2}=\frac{\partial}{\partial y}\left(2y\frac{dg}{du}\right)=2\frac{dg}{du}+2y\frac{\partial}{\partial y}\left(\frac{dg}{du}\right)=2\frac{dg}{du}+2y\left(\frac{d^2g}{du^2}\frac{\partial u}{\partial y}\right)=2g'(u)+4y^2g''(u)$$

Combining these we get:

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=4g'(u)+4(x^2+y^2)g''(u)=4g'(u)+4u\,g''(u)\text{, where } u = x^2 + y^2.$$

(4) (14 pts) Cartesian and polar coordinates are related by $\left\{\begin{matrix} x=r\cos\theta \\ y=r\sin\theta \end{matrix}\right\}$. Suppose that at a certain instant in time, a particle is located at $(x, y) = (3, 4)$, and its polar coordinates are changing as specified by $\frac{dr}{dt}=2$ and $\frac{d\theta}{dt}=1$. Use the chain rule to calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for the particle at this instant.

Solution: As in the previous problem, it's important to understand the variables and the dependencies. In this case, we have a situation where $t \to (r,\theta) \to (x, y)$. Applying the Chain Rule, we have:

$$\frac{dx}{dt}=\frac{\partial x}{\partial r}\frac{dr}{dt}+\frac{\partial x}{\partial \theta}\frac{d\theta}{dt}=\cos\theta\frac{dr}{dt}-r\sin\theta\frac{d\theta}{dt}=\left(\frac{3}{5}\right)2-5\left(\frac{4}{5}\right)1=-\frac{14}{5}$$
$$\frac{dy}{dt}=\frac{\partial y}{\partial r}\frac{dr}{dt}+\frac{\partial y}{\partial \theta}\frac{d\theta}{dt}=\sin\theta\frac{dr}{dt}+r\cos\theta\frac{d\theta}{dt}=\left(\frac{4}{5}\right)2+5\left(\frac{3}{5}\right)1=\frac{23}{5}$$

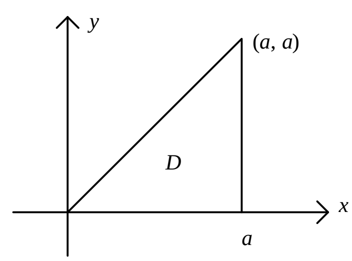
Here we use the fact that when $(x, y) = (3, 4)$, we'll have a 3-4-5 triangle from which the radius r and the cosine and sine of the angle θ are easily computed.

(5) (14 pts) **(a)** Using Cartesian coordinates, evaluate the integral of the function $x^2 + y^2$ over the right triangle with vertices $(x, y) = (0, 0)$, $(a, 0)$, and (a, a).

Solution: $\iint_D (x^2+y^2)\,dA=\int_0^a\int_0^x (x^2+y^2)\,dydx$.

The inner integral gives $\left[x^2y+\tfrac{1}{3}y^3\right]\Big|_{y=0}^{y=x}=x^3+\tfrac{1}{3}x^3=\tfrac{4}{3}x^3$.

The outer integral gives $\int_0^a \tfrac{4}{3}x^3dx=\tfrac{1}{3}x^4\Big|_0^a=\tfrac{1}{3}a^4$



(5b) Evaluate the same integral using polar coordinates. (Hint: make the substitution $u=\tan\theta$.)

Solution: If we use polar coordinates, the integrand function becomes r^2 and we must use $dA = rdrd\theta$. The θ limits for the integral are easy, but for any given θ, the variable r will go from $r = 0$ (the origin) out to the line where $x = 1$. In polar coordinates this gives $r\cos\theta=1$ or $r=\sec\theta$. We thus get:

$$\iint_D (x^2+y^2)\,dA=\int_0^{\pi/4}\int_0^{a\sec\theta} r^2\,rdrd\theta=\int_0^{\pi/4}\int_0^{a\sec\theta} r^3\,drd\theta=\tfrac{1}{4}a^4\int_0^{\pi/4}\sec^4\theta\,d\theta$$
$$=\tfrac{1}{4}a^4\int_0^{\pi/4}\sec^2\theta\sec^2\theta\,d\theta=\tfrac{1}{4}a^4\int_0^{\pi/4}(1+\tan^2\theta)\sec^2\theta\,d\theta=\tfrac{1}{4}a^4[\tan\theta+\tfrac{1}{3}\tan^3\theta]\Big|_0^{\pi/4}=\tfrac{1}{4}a^4\,\tfrac{4}{3}=\tfrac{1}{3}a^4$$

(6) (14 pts) Convert the integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}dydx$ to polar coordinates and hence evaluate it exactly. Sketch the region R over which the integration is being performed.

Solution: The region for this integral is a semicircle of radius 2 centered at the origin and with $y \geq 0$. If we switch to polar coordinates, the integral becomes $\int_0^{\pi}\int_0^2 e^{-r^2} rdrd\theta$. The inner integral gives $-\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=2} = -\frac{1}{2}(e^{-4}-1) = \frac{1}{2}\left(1-\frac{1}{e^4}\right)$. The outer integral then gives us $\frac{\pi}{2}\left(1-\frac{1}{e^4}\right)$

(7) (14 pts) Suppose that a mass density function is given by $\delta(x, y, z) = x + z$. Set up, <u>but do not evaluate</u>, an iterated integral for the mass of the body which has this density function and which is bounded by the surfaces $x^2 + y^2 = 4$, $x + y + z = 5$, and $z = 1$.

Solution: Here we use the fact that with this density function the mass will be given by $\iiint_B \sigma(x, y, z)dV$. Perhaps the trickiest part of the problem is the fact that although the solid region lies within the cylinder $x^2 + y^2 = 4$, you must conclude that the bottom of the region is $z = 1$ and the top is given by the plane $x + y + z = 5$ or, more appropriately, $z = 5 - x - y = 5 - r\cos\theta - r\sin\theta$ if we choose to use cylindrical coordinates. We'll also have to write the density in cylindrical coordinates. This gives us the integral:

$$\int_0^{2\pi}\int_0^2\int_1^{5-r\cos\theta-r\sin\theta}(r\cos\theta + z)\, rdzdrd\theta$$

Though Cartesian coordinates don't lend themselves as well to this integral, should you choose to use them you'll get the integral (doing the x integration last):

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_1^{5-x-y}(x+z)\, dzdydx$$