Problems that didn't make it to the Final Exam (but which may still provide useful practice!)
[These problems come without warranty.]

(1) Ricky the Rodent, Harvard's most experienced laboratory rat, finally learns his way out of the Psych Labs and takes up residence on the third floor of the Science Center, where he learns trigonometry. But old habits die hard, and even in freedom Ricky feels compelled to run around in complicated closed paths. His favorite is $\mathbf{X}(t) = \cos t\,\mathbf{i} + \sin 2t\,\mathbf{j}$.

(a) On the diagram, indicate Ricky's position at times $t = \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.

(b) With what speed does Ricky pass through the point (0,0)?
(c) On the diagram, sketch Ricky's velocity vectors the first time he passes through the origin (call it $\mathbf{v}_1$) and the second time (call it $\mathbf{v}_2$). What is the cosine of the angle θ between these vectors?
(d) Write down a definite integral whose value equals the total distance that Ricky covers between $t = 0$ and $t = 2\pi$. [Do not attempt to evaluate it!]

(2) When an object that lies entirely in the xy-plane rotates about the z-axis, the velocity of every point on the object is described by the vector field $\mathbf{v} = A\mathbf{k} \times \mathbf{r}$, where $\mathbf{r}$ is the position vector of the point: $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ and A is the rate of rotation. Calculate the divergence and curl of this vector field.

(3) Define the vector field $\mathbf{F}$ by $\mathbf{F} = 2xy\,\mathbf{i} + x^2\,\mathbf{j}$.
(a) Parametrize the curve C, which is the portion of the hyperbola $xy = 4$ that starts at $x = 1$, $y = 4$ and ends at $x = 4$, $y = 1$, using a parameter t over the interval from 1/2 to 2.
(b) Using this parametrization, evaluate the line integral of $\mathbf{F}$ along the curve C.
(c) Find a function g whose gradient equals $\mathbf{F}$, and use it to determine the value of the same line integral.

(4) Find the minimum value of the function $f(x, y) = \frac{2}{x} + \frac{4}{y} + xy$ for points (x, y) in the first quadrant.

(5) Coordinate axes are drawn, centered on the mythical planet Marz, with length in units such that the planet's surface is given in spherical coordinates by $\rho = 1$. At this very moment, the surface temperature is given in terms of the spherical coordinates θ and ϕ by $T(\theta, \phi) = 130\sin\phi + 15\sin\phi\sin\theta - 40$.
(a) The region of points on the surface of Marz that are in daytime are given by the points where $0 \le \theta \le \pi$. Find the average current temperature over this region of the planet.
(b) Marz' northern polar ice cap is given by the set of points on the planet's surface where $0 \le \phi \le \pi/6$. Find the average current temperature on the nighttime portion of this ice cap.

(6) Suppose the surface of a planet can be described by the equation $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$.

If the temperature at any point on this surface is given by $T(x, y, z) = 2x^2 + yz - 4z + 150$, find the hottest and coldest spots on the surface of the planet.

(7) Calculate $\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$ where R is the region bounded by the hyperbolas $xy = 1$ and $xy = 9$ and by the lines $y = x$ and $y = 4x$. Use the coordinate transformation $x = \frac{u}{v}$, $y = uv$.

(8) Find the circulation of $\mathbf{F} = (-y, x)$ along the closed curve $|x| + |y| = 1$. (This is a square, by the way.)

(9) Let R be the region of the plane bounded by the x-axis and the parabola $y = 1 - x^2$.

(a) Evaluate the integral of the function $f(x, y) = y$ over the region R.

Now consider coordinates (u, v) in the plane that are related to Cartesian coordinates (x, y) by the equations $x = uv$, $y = v^2 - u^2$.

(b) Sketch the region R in the xy-plane, and sketch the triangle in the uv-plane bounded by the lines $u = v$, $u = -v$, and $v = 1$. Explain precisely how the sides of this triangle correspond to the boundary of R.

(c) Write down the Jacobian matrix for this change of variables, and show that its determinant equals $2(v^2 + u^2)$.

(d) Evaluate the same integral as in part (a) in terms of the coordinates u and v.

(10) A sphere is described by $x^2 + y^2 + z^2 = a^2$.

(a) Write down a quotient of two surface integrals (note -- these are not flux integrals) that equals the average value of $x^2 + y^2$ over this spherical surface.

(b) Write down a parametrization for this sphere, expressing x, y, and z in terms of the usual spherical coordinates ϕ and θ. By evaluating the cross product of appropriate vectors formed from partial derivatives, show that $dS = a^2 \sin\phi \, d\phi d\theta$.

(c) Use the parametrization in (b) to evaluate the integrals in a) and thereby show that the average value of $x^2 + y^2$ over the spherical surface is 2/3.

(11) The pressure in the ocean is a function of the depth z: $P(z) = P_0 + wz$, where P_0 is a constant (atmospheric pressure) and w is the weight per unit volume of sea water.

On a submerged object like a submarine, each small portion of the surface (area ΔS) experiences a pressure force of magnitude $P(z)\Delta S$, directed along the inward normal $-\mathbf{n}$.

(a) Write down a Riemann sum that approximates the total pressure force $\mathbf{F}$ (a vector) on a submerged object.

(b) The z-component F_z of this total pressure force is just a number. Write down a flux integral over the surface of the submerged object that equals F_z.

(c) Use the divergence theorem to convert this flux integral to a volume integral.

(d) By similar reasoning, determine the other components F_x and F_y of the total pressure force.

(e) Now that you have all the components of the total pressure force, explain why Archimedes was correct in his statement that “the total pressure force on a submerged object is equal and opposite to the weight of the fluid displaced by the object.”

(12) A region R in the plane is bounded by the x-axis, the y-axis, and the quarter-circle $y = \sqrt{1 - x^2}$. Let $\mathbf{F}$ be the vector field $\mathbf{F} = x(1 - y^2)\mathbf{i}$.

(a) Evaluate the line integral of $\mathbf{F}$ counterclockwise around the boundary of R.

(b) Use Green’s Theorem to find a double integral that has the same value as this line integral, and then evaluate it using polar coordinates.

(13) Compute the integral $\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} \, dx dy dz$.