Math 21a - Spring 2001 - Problem Set #1

It must be emphasized that the suggested practice problems in each section of the text are an essential part of your preparation for these problem sets. However, only the problem sets need be submitted for credit. The course assistant for your section can also check your work on the daily assignments if you wish.

Key topics: $\mathbf{R}^2$ and $\mathbf{R}^3$, points vs. vectors, addition and scaling of vectors, difference vector, norm (length) of a vector, distance between points, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ notation, unit vectors. Examples of curves and surfaces defined by algebraic equations: lines, circles, spheres, cylinders. Parametrized curves in $\mathbf{R}^2$ and $\mathbf{R}^3$, parametrization of lines and a line segments, circles, spirals, and helices. Differentiation of vector valued functions, velocity and acceleration vectors, speed, arclength. Dot product in $\mathbf{R}^2$, $\mathbf{R}^3$, and $\mathbf{R}^n$. Scalar and vector projections. Equation of a plane in $\mathbf{R}^3$ from a point and a normal vector. Relationship between the dot product of vectors, lengths, angle between vectors. Perpendicularity. Distance from a point to a line or a plane. Differentiation formulas involving the dot product.

Problem 1 (The Pythagorean Theorem): Everyone knows it, but do you know why it's true? Either prove it or find a proof of it from another source. Write down the argument and submit it.

Problem 2 (The Law of Cosines): Using the Pythagorean Theorem (or something else, if you prefer), derive the Law of Cosines. Make sure that your argument is valid for any triangle and that all possible cases are covered. [Note: Since we derived the formula $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ using the Law of Cosines, it would be circular to use this to derive the Law of Cosines.]

Problem 3: Let P denote the plane where $2x - 2y + z = 3$.

(a) Find three points in P which do not all lie on the same line.
(b) Find a vector that is perpendicular to P.
(c) Find the distance from P to the origin.
(d) Write down an equation for a line that lies entirely in P.

Problem 4: A particle moving in space has position at time t given by $(3\sin t^2, 3\cos t^2, 4t^2)$.

(a) Find the coordinates of the particle at $t = \sqrt{\pi}$.
(b) Find the velocity vector of the particle at $t = \sqrt{\pi}$.
(c) Find parametric equations for the line tangent to the trajectory at $t = \sqrt{\pi}$.
(d) Find the distance traveled by the particle between $t = 0$ and $t = \sqrt{\pi}$.

Problem 5:

(a) Give parametric equation(s) for the line of intersection of the planes $2x + 2y - z = 15$ and $5x + 3y - 3z = 32$.
(b) What angle does the line in a) make with the positive z-axis?
(c) Find the point on the plane $2x + 2y - z = 15$ closest to the point (-1,1,3).

Problem 6:

Let M = (2, 0, 0), N = (3, 3, -1), P = (-1, -2, 1), Q = (1, 3, 7) be points in $\mathbf{R}^3$.

(a) Find parametric equations for the line containing M and Q.
(b) Find an equation for the plane containing M, N, and P.
(c) Find the area of the triangle MNP.
(d) Find the volume of the pyramid MNPQ.

Problem 7: Show that if **a**, **b**, **c**, and **d** are positions of the vertices of a quadrilateral (four-sided figure), not necessarily lying in a plane, then the midpoints of the four edges are vertices of a parallelogram.

Problem 8 (from last semester's midterm): In the literary classic Moby Dick by Herman Melville, the captain of the whaling ship Pequod became obsessed with a particular white whale. The ship's log briefly described an encounter with this whale as follows: "In a coordinate system where the Pequod is at the origin, the ocean surface is the plane $z = 0$ and $\mathbf{k} = (0,0,1)$ points up. The whale swam on a straight line until it reached the surface; this line passed through the point (-2,8,-6) at time $t = 0$ and through the point (1,6,-4) at $t = 1$." No more about the encounter appeared in the log. Given the information just provided, answer the following questions:

(a) Find parametric equation(s) for the line that the whale traced as it swam along.

(b) As you know, whales are mammals so must surface to breathe. What are the coordinates of the point where the whale surfaced for air?

(c) What is the closest distance that the whale came to the Pequod?

Problem 9 (from last semester's midterm): Suppose that **u** and **v** are both non-zero vectors in space and set $\mathbf{w} = \|\mathbf{u}\|\,\mathbf{v} - \|\mathbf{v}\|\,\mathbf{u}$. Answer the following questions either true or false and justify your answer.

(a) The vector **w** is zero only when $\mathbf{u} = \mathbf{v}$.

(b) **w** is perpendicular to $\|\mathbf{u}\|\,\mathbf{v} + \|\mathbf{v}\|\,\mathbf{u}$.

(c) Assuming that **w** is not zero, then **w** is never parallel to either the x, y, or z coordinate axes.

Problem 10 (from Fall 99 midterm):

(a) Let L be a line in $\mathbf{R}^3$ passing through the origin and parallel to the vector $\mathbf{v} = \mathbf{j} + \mathbf{k}$, i.e. $\mathbf{v} = (0, 1, 1)$. Let $P = (x, y, z)$ be a point that is not on the line L. Find an expression in terms of x, y, and z for the (shortest) distance from P to L.

(b) Find an equation for the radius 1 cylinder with the line L as its central axis and with a radius of 1. (Hint: What geometric property defines a cylinder?)

(c) Give sketches of an $x = 0$ and a $y = 0$ cross section of the cylinder.

Challenge problem:

In a game of baseball, a batted ball leaves home plate at ground level, moving without air resistance. At the precise instant that the ball leaves home plate, an outfielder begins to run with constant velocity toward the place where the ball will land, with speed carefully chosen to arrive there at the same time as the ball. Show that the tangent of the angle of elevation of the ball above the horizon, as seen by the running outfielder, increases linearly as a function of time.

Simpler version: Do the case where the outfielder is already standing where the ball will land.
(By relativity, this is equivalent to the general problem.)