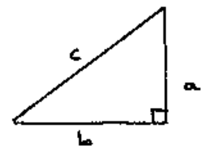
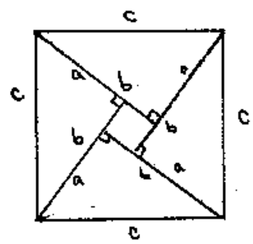
P

Math 21a Solutions to Problem Set # 1

① Consider the right triangle



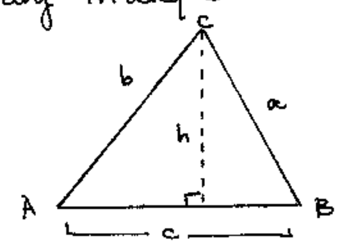
and construct the following square:



$$\Rightarrow c^2 = 4\left(\tfrac{1}{2}ab\right) + (b-a)^2$$
$$= 2ab + b^2 + a^2 - 2ab$$
$$= a^2 + b^2$$

(note: there are a number of other solutions to this problem)

② Consider an arbitrary triangle ABC and drawn in one of its heights, h:



(note: this proof will not depend on whether or not any of these angles is acute or obtuse, so it is valid in all cases)

Projecting AC and BC onto the height h yields
$$h = a \sin B = b \sin A$$

By projecting AC and BC onto AB, we get
$$c = a\cos B + b\cos A$$
$$\rightarrow \cos^2 = a^2\cos^2 B + b^2\cos^2 A + 2ab\cos A\cos B$$
$$= a^2 + b^2 - a^2\sin^2 B - b^2\sin^2 A + 2ab\cos A\cos B \quad \text{(apply first equality)}$$
$$= a^2 + b^2 - a\sin B\, b\sin A - b\sin A\, a\sin B + 2ab\cos A\cos B$$
$$= a^2 + b^2 + 2ab(\cos A\cos B - \sin A\sin B)$$
$$= a^2 + b^2 + 2ab\cos(A+B)$$
$$= a^2 + b^2 + 2ab\cos(\pi - C) = a^2 + b^2 - 2ab\cos C$$

3a) $(0,0,3)$, $(1,0,1)$, and $(0,-1,1)$ are three noncolinear points in P.

3b) The vector $(2,-2,1)$ is normal (perpendicular) to P.

3c) For what scalar k does $k(2,-2,1) = (2k,-2k,k)$ touch P?
$$2(2k) - 2(-2k) + k = 9k = 3 \longrightarrow k = \tfrac{1}{3}$$
So the distance from P to the origin is the length of $\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$
$$\sqrt{\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{-2}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2} = \sqrt{\tfrac{4+4+1}{9}} = 1$$

(3d) The line connecting $(0,0,3)$ and $(1,0,1)$ will lie entirely within P.

We can parametrize this line by

$$x = t, \quad y = 0, \quad z = 3 - 2t$$

(4a) At $t = \sqrt{\pi}$, what is $\vec{x}(t)$?

$$\vec{x}(\sqrt{\pi}) = (3\sin\pi,\ 3\cos\pi,\ 4\pi) = (0, -3, 4\pi)$$

(4b) $v(t) = \vec{x}\,'(t) = \left((3\cos t^2)(2t),\ (-3\sin t^2)(2t),\ 8t\right)$

$$v(\sqrt{\pi}) = \left((3\cos\pi)(2\sqrt{\pi}),\ (-3\sin\pi)(2\sqrt{\pi}),\ 8\sqrt{\pi}\right)$$

$$= (-6\sqrt{\pi},\ 0,\ 8\sqrt{\pi})$$

(4c) $v(\sqrt{\pi})$ is the direction vector for the point $\vec{x}(\sqrt{\pi})$, so parametric equations for the line tangent to the trajectory at $t = \sqrt{\pi}$ are

$$x = -6\sqrt{\pi}\,t, \quad y = -3, \quad z = 4\pi + 8\sqrt{\pi}\,t$$

(4d)

$$\text{distance} = \int_0^{\sqrt{\pi}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{\sqrt{\pi}} \sqrt{(-6\sqrt{\pi})^2 + 0 + (8\sqrt{\pi})^2}\,dt = \int_0^{\sqrt{\pi}} \sqrt{100\pi}\,dt = 10\sqrt{\pi}\int_0^{\sqrt{\pi}} dt = 10\pi$$

(5a)

$$2x + 2y - z = 15 \longrightarrow z = 2x + 2y - 15$$

$$5x + 3y - 3z = 32 \longrightarrow 3z = 5x + 3y - 32$$

$$\longrightarrow 5x + 3y - 32 = 6x + 6y - 45$$

$$\longrightarrow x = -3y + 13$$

Plug this back into the first equation to get

$$z = -6y + 26 + 2y - 15 = -4y + 11$$

So the intersection of the plane can be parametrized by

$$x = 3t + 13, \quad y = -t, \quad z = 4t + 11$$

(5b) This line has direction vector $v = (3, -1, 4)$. The z-axis (positive) has direction vector $w = (0, 0, 1)$. Find the angle θ between them by theorem 5 on page 66 of the text:

$$\cos\theta = \frac{v \cdot w}{|v||w|} = \frac{0(3) + 0(-1) + 1(4)}{\sqrt{9+1+16}\,\sqrt{0+0+1}} = \frac{4}{\sqrt{26}}$$

So $\theta = \cos^{-1}\left(\frac{4}{\sqrt{26}}\right) \approx 0.6690$ rad

(5c) To find the closest point we would need to draw a perpendicular from (-1, 1, 3) to the plane. The normal vector gives the direction of this perpendicular, so we need to find the value of k such that

$$(-1,1,3)+k(2,2,-1) = (-1+2k, 1+2k, 3-k)$$

is in the plane...

$$2(-1+2k)+2(1+2k)-(3-k) = -2+4k+2+4k-3+k = 9k-3 = 15$$

$$\rightarrow k = 2$$

Thus the closest point in the plane to (-1, 1, 3) is

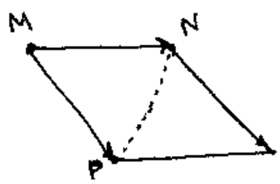
$$(-1,1,3)+2(2,2,-1) = (3,5,1)$$

(6a) $x = 2-t,\ y = 3t,\ z = 7t$ is a line containing M and Q.

(6b) The vector $\overrightarrow{MN}$ is (1, 3, -1) and the vector $\overrightarrow{MP}$ is (-3, -2, 1) so the normal vector to this plane is $\overrightarrow{MN} \times \overrightarrow{MP} = (1,2,7)$. Thus an equation for the plane containing M, N and P is

$$x + 2y + 7z = 2$$

(6c)



The area of triangle MNP is ½ the area of the parallelogram formed by the two vectors. And this has area $|\overrightarrow{MN} \times \overrightarrow{MP}| = |(1,2,7)| = \sqrt{1^2+2^2+7^2} = \sqrt{54} = 3\sqrt{6}$

So area of triangle MNP is $\frac{3}{2}\sqrt{6}$.

(6d) This is $(\overrightarrow{MN} \times \overrightarrow{MP}) \cdot \overrightarrow{MQ}$

$$= (1,2,7)\cdot(-1,3,7) = -1+6+49 = 54$$

(7) If a, b, c, d are the positions of the vertices, then the midpoints are at:

$$M_1 = \frac{a+b}{2},\ M_2 = \frac{b+c}{2},\ M_3 = \frac{c+d}{2},\ M_4 = \frac{a+d}{2}$$ (assuming the a, b, c, d are in order)

$\overrightarrow{M_1M_2} = \frac{c-a}{2}$ and $\overrightarrow{M_3M_4} = \frac{a-c}{2}$ so these two sides are parallel.

$\overrightarrow{M_1M_4} = \frac{d-b}{2}$ and $\overrightarrow{M_2M_3} = \frac{d-b}{2}$ so these two sides are parallel.

Thus $M_1M_2M_3M_4$ is a parallelogram.

(8a) $x = -2 + 3t$, $y = 8 - 2t$, $z = -6 + 2t$

(8b) $z = 0 \rightarrow t = 3$. So the whale surfaced at $(7, 2, 0)$

(8c) The distance from $(0,0,0)$ to a generic point $(-2+3t, 8-2t, -6+2t)$ on the line

$$d = \sqrt{[(-2+3t)^2 + (8-2t)^2 + (-6+2t)^2]} = \sqrt{[17t^2 - 68t + 104]}$$

$$d^2 = 17t^2 - 68t + 104$$

minimizing d^2 with respect to t . . .

$$(d^2)' = 34t - 68 = 0 \rightarrow t = 2 \text{ (check that this is a minimum)}$$

So the closest distance is

$$\sqrt{68 - 136 + 104} = \sqrt{36} = 6 \text{ units.}$$

(10a) Parametrizing L gives $x = 0$, $y = t$, $z = t$.

To find the shortest distance we must minimize

Sorry about the wrong order!

$$D(t) = \sqrt{x^2 + (y-t)^2 + (z-t)^2} = \sqrt{x^2 + y^2 + z^2 + 2t^2 + 2(y+z)(-t)}$$

$$(D^2)' = 4t - 2(y+z) = 0$$

$$\rightarrow t = \tfrac{1}{2}(y+z)$$

So the minimum distance is

$$\sqrt{x^2 + y^2 + z^2 + \tfrac{1}{2}(y+z)^2 - (y+z)(y+z)}$$

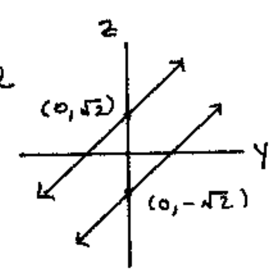
$$= \sqrt{x^2 + y^2 + z^2 - \tfrac{1}{2}(y^2 + z^2 + 2yz)} = \sqrt{x^2 + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2 - yz} = \sqrt{x^2 + \tfrac{1}{2}(y-z)^2}$$

(10b) For any point $P = (x, y, z)$ in this cylinder, the distance from P to L is 1. So by (10a)

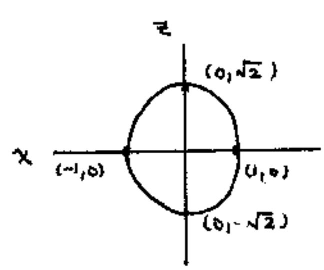
$x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 - yz = 1$ is an equation for this cylinder.

(10c) $x = 0 \rightarrow y^2 + z^2 - 2yz = (y-z)^2 = 2$

$\rightarrow y + \sqrt{2} = z$ and $y - \sqrt{2} = z$



$y = 0 \rightarrow x^2 + \frac{1}{2}z^2 = 1$



(9a) u and v nonzero vectors. $w = \|u\|v - \|v\|u$.

w is zero only when $u = v$?

Say $w = 0$. So $\|u\| v = \|v\| u$. Then $v = \frac{u}{\|u\|} \cdot \|v\|$.

$\frac{u}{\|u\|}$ is a unit vector so $\frac{u}{\|u\|} \cdot \|v\|$ is a vector the length of v in the direction of u. So u and v have the same direction. However, let $u = (2,0,0)$ and $v = (1,0,0)$ and notice that $\|u\|v - \|v\|u = 0$. So the statement IS FALSE.

(9b) w is perpendicular to $\|u\|v + \|v\|u$?

$$w \cdot (\|u\|v + \|v\|u) = \|u\|^2 (v \cdot v) + \|u\|\|v\| (u \cdot v) - \|u\|\|v\| (u \cdot v) - \|v\|^2 (u \cdot u)$$
$$= \|u\|^2 \|v\|^2 - \|v\|^2 \|u\|^2 = 0$$

So the vectors are perpendicular and the statement is TRUE.

(9c) If $w \neq 0$ then w is never parallel to either the x, y, z coordinate axes?

Let $u = (1,1,0)$ and $v = (1,-1,0)$.

Then $w = \sqrt{2}\,(1,-1,0) - \sqrt{2}\,(1,1,0) = (0, -2\sqrt{2}, 0)$.

Thus we have a nonzero w parallel to the y-axis so the statement is false.