Math 21a - Spring 2001 - Problem Set #2

Dates covered: (Feb 12 - Feb 23 - two weeks)
Text sections: 1.5-1.8, 2.1, and selected topics in 5.1 and 5.2
Key topics: Lines and planes and their intersection. The cross product in $\mathbf{R}^3$ and its geometric and algebraic properties. Use of the cross product in finding normal vectors, the area of a parallelogram, and the volume of a parallelepiped (triple scalar product). Further topics in parametrized curves. Functions of several variables with emphasis on functions of two and three variables. Graphs, contours, and level sets. Integration of a function along a parametrized curve. Introduction to vector fields and the work done by a variable force along a parametrized curve.

Mathematica Lab #1 (posted on the Math 21a website) may be helpful in optional parts which ask you to draw parametrized curves. You can cut and paste the graphs into a Word document, if you like. Mathematica Lab #2 may prove helpful in understanding some things about graphs and level sets.

Problem 1: Prove the Law of Sines by considering how to calculate the area of a triangle from vectors along any two of the three sides. [The Law of Sines says that if the triangle has angles A, B, and C and if the sides opposite those angles are, respectively, a, b, and c, then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.]

Problem 2: Consider a tetrahedron defined by three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ leading out of one vertex. For each face, define a normal vector, pointing out of the tetrahedron, whose length equals the area of the face. Show that these four normal vectors sum to the zero vector.

Problem 3: A bug is flying through the air and its position at time t is given by the end point of the vector $\mathbf{r}(t) = (2\ln t, t^2, -2\sqrt{2}\,t)$.

(a) What is the bug's velocity at time t?
(b) What is the bug's speed at time t?
(c) What is the length of the path of the bug between $t = 1$ and $t = 2$?
(d) What is the component (vector projection) of the bug's position vector at time 1 in the direction of the vector (1,1,1)? (Express your answer as a vector).

Problem 4: A bicycle wheel of radius 1 foot rolls to the right along the horizontal x-axis without slipping with a (horizontal) speed of π feet per second. Suppose a bug initially at the hub of the wheel starts to crawl outward along a spoke just as the wheel passes over the origin (at $t = 0$) and that the bug moves at a rate of 1 inch every second along the spoke. (Assume the bug moves initially downward.)

(a) Write down a parametrization describing the bug's motion.

(b) Give a good sketch of the motion of the bug. (A graphing calculator will do it in parameter mode.) (Optional: Use Mathematica or a comparable tool to make your graph.)

(c) Find the velocity vector of the bug 6 seconds into its brief adventure.

(d) Find the bug's speed (in space) at the 6 second mark.

(e) Write down an integral giving the total distance the bug travels in space during its journey to the edge of the wheel. [Even after simplifying the answer will dreadful!]

(f) (Optional) Find the numerical value of this distance. [Don't look for antiderivatives!]

Problem 5: Ostebee-Zorn problems 1.6/16,22 and 1.7/30

Problem 6: Ostebee-Zorn 2.1/6,12.

Problem 7: Let $f(x, y) = x^2 - y^3$ and consider what happens as we pass through the point (2, 1) in various ways.

(a) Find a parametrization $\mathbf{x}(t)$ for the horizontal line (parallel to x-axis) where you move in the positive x direction with a constant speed of 1 unit/sec and where $\mathbf{x}(0) = (2,1)$. Calculate the rate of change at $t = 0$ of the values of f as you move along this path. That is, find the value of $\frac{d}{dt} f(\mathbf{x}(t))$ at $t = 0$.

(b) Do the same construction as in (a), only with the parametrized line passing upward through (2,1), i.e. parallel to the y-axis, and again with unit speed and $\mathbf{x}(0) = (2,1)$.

(c) Do the same construction one more time, only along the line through (2,1) with slope 2, moving upward to the right, and again with unit speed and $\mathbf{x}(0) = (2,1)$.

Problem 8: Repeat the previous problem with the function $f(x, y, z) = 3x^2 + 5y - 7z^2$, the point (1,1,1), and the directions: (a) positive x direction, (b) positive y direction, (c) positive z direction, (d) along a line in the direction of the vector (3,4,12). Again, in all cases move at unit speed.

Problem 9: Let $\mathbf{F} = (2xy^2 + 3x^2, 2x^2y)$. Compute the line integral (work) of $\mathbf{F}$ from (0, 0) to (1, 1) along the following paths:

a) The diagonal $x = y$.
b) Along the x-axis from $x = 0$ to $x = 1$, then from (1, 0) to (1, 1) along the line $x = 1$.
c) Along the y-axis from $y = 0$ to y = 1, then from (0, 1) to (1, 1) along the line $y = 1$.

Problem 10: Let $\mathbf{F} = (xy, y^2 - z, 3y)$. Compute the line integral (work) of $\mathbf{F}$ along the line segment from the origin to the point (1, 1, 1).