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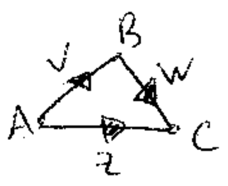
MATH 21A PS 2 SOLUTIONS

1) Given a triangle, ΔABC formed by vectors v, z, w, we know that the area of ΔABC is given by $\frac{1}{2}|v \times w| = \frac{1}{2}|v \times z| = \frac{1}{2}|w \times z|$. Longhand,



this is $|v||w|\sin B = |v||z|\sin A = |w||z|\sin C$

Now divide through to obtain

$$\frac{\sin B}{|z|} = \frac{\sin A}{|w|} = \frac{\sin C}{|v|},$$ the law of sines. ■

2) Let T be the tetrahedron formed by vectors v_1, v_2, v_3:

Let $w_1 = v_3 - v_1$
$w_2 = v_3 - v_2$

Let F_1 be the face spanned by v_1, v_2, F_2 by v_1, v_3, F_3 by w_1, w_2, and F_4 by v_2, v_3. Using the right-hand rule, if n_i be the normal vector whose length is the area of F_i (pointing outwards of the face F_i) Then one has:

$$n_1 = v_2 \times v_1 \quad n_2 = v_1 \times v_3, \quad n_3 = w_1 \times w_2, \quad n_4 = v_3 \times v_2$$

Then

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 &= v_2 \times v_1 + v_1 \times v_3 + v_3 \times v_2 + (v_3 - v_1) \times (v_3 - v_2) \\ &= v_2 \times v_1 + v_1 \times v_3 + v_3 \times v_2 + v_3 \times v_3 - v_3 \times v_2 - v_1 \times v_3 + v_1 \times \\ &= 0 \quad (\text{using } v_1 \times v_2 = -v_2 \times v_1). \end{aligned}$$

■

3) a) $V(t) = r'(t) = \left(\frac{2}{t}, 2t, -2\sqrt{2}\right)$

b) $s(t) = |V(t)| = \sqrt{\frac{4}{t^2} + 4t^2 + 8} = 2\sqrt{\frac{1}{t^2} + t^2 + 2} = 2\sqrt{\left(t + \frac{1}{t}\right)^2}$

$= \boxed{2\left|t + \frac{1}{t}\right|}$

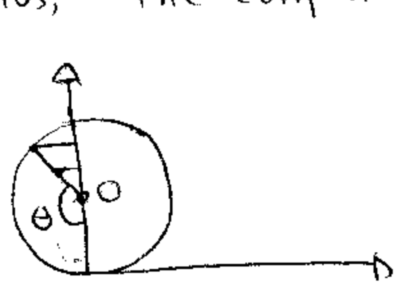
c) $l = \int_1^2 |V(t)|\,dt = 2\int_1^2 \left(t + \frac{1}{t}\right)dt = 2\left[\frac{t^2}{2} + \log t\right]_1^2$

$= 2\left(2 + \log 2 - \frac{1}{2}\right) = \boxed{3 + 2\log 2}$

d) $r(1) = (0, 1, -2\sqrt{2})$ The magnitude of the component is $\frac{(1,1,1)\cdot(0,1,-2\sqrt{2})}{\sqrt{3}} = \frac{1-2\sqrt{2}}{\sqrt{3}}$

Thus, the component is $\frac{1-2\sqrt{2}}{\sqrt{3}} \frac{(1,1,1)}{\sqrt{3}} = \frac{1-2\sqrt{2}}{3}(1,1,1)$

4) a)



$r = 12$ in

$v_o = 12\pi$ in/sec

$v_{bug} = 1$ in/sec

So the radius of the bug at time t is $r_{bug} = t$.

θ at time t is given by $\theta = \pi t$ (since in 1 sec, θ traverses π rads so that the wheel has moved $\pi r = 12\pi$ in)

The position of the bug is then simple to calculate:

$\boxed{x(t) = 12\pi t - t\sin(\pi - \theta) = 12\pi t - t\sin(\pi t)}$

eg. $\boxed{r(t) = (12\pi t - t\sin(\pi t), 12 - t\cos}$

$\boxed{y(t) = 12 + t\cos(\pi - \theta) = 12 - t\cos(\pi t)}$

b) See attached sheet.

c) $V(t) = r'(t) = (12\pi - \pi t\cos(\pi t) - \sin(\pi t),\ \pi t\sin(\pi t) - \cos(\pi t))$

$V(6) = (12\pi - 6\pi, -1) = \boxed{(6\pi, -1)}$

d) $s(6) = |V(6)| = \sqrt{1+36\pi^2} \approx 18.8$ in/sec

e) $\int_0^{12} |V(t)|\,dt = \int_0^{12} \sqrt{(12\pi - \pi t\cos(\pi t) - \sin(\pi t))^2 + (\pi t\sin(\pi t) - \cos(\pi t))^2}$

$= \int_0^{12} \sqrt{1+144\pi^2+\pi^2t^2-24\pi^2 t\cos(\pi t)-24\pi\sin(\pi t)}\,dt$

f) see attached sheet $\boxed{492.232 \text{ in}}$

5) 16) $\frac{d}{dt}\left(\frac{r}{|r|}\right) = \frac{d}{dt}\frac{r}{(r\cdot r)^{1/2}} = \frac{d}{dt}(r\cdot r)^{-1/2} r$

$= (r\cdot r)^{-1/2} r' - \frac{1}{2}(r\cdot r)^{-3/2}(2r\cdot r')r$

$= (r\cdot r)^{-1/2} r' - (r\cdot r)^{-3/2}(r\cdot r')r$

$= \frac{r'}{|r|} - \frac{r\cdot r'}{|r|^3} r$

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In[5]:= ParametricPlot[{12 Pi t - t Sin[Pi t], 12 - t Cos[Pi t]}, {t, 0, 12}]
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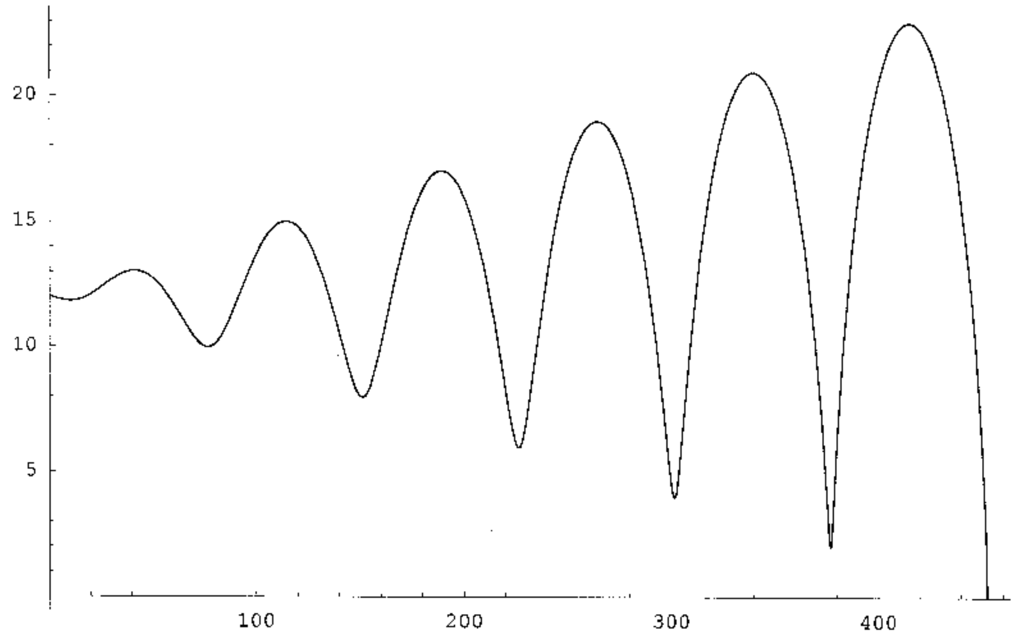


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Out[5]= - Graphics -

In[14]:= Expand[(12 Pi - Pi t Cos[Pi t] - Sin[Pi t])^2 + (Pi t Sin[Pi t] - Cos[Pi t])^2]
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Out[14]= $144\pi^2 - 24\pi^2 t \,\mathrm{Cos}[\pi t] + \mathrm{Cos}[\pi t]^2 + \pi^2 t^2 \,\mathrm{Cos}[\pi t]^2 - 24\pi \,\mathrm{Sin}[\pi t] + \mathrm{Sin}[\pi t]^2 + \pi^2 t^2 \,\mathrm{Sin}[\pi t]^2$

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In[15]:= TrigReduce[%]
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Out[15]= $1 + 144\pi^2 + \pi^2 t^2 - 24\pi^2 t \,\mathrm{Cos}[\pi t] - 24\pi \,\mathrm{Sin}[\pi t]$

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In[16]:= NIntegrate[Sqrt[%], {t, 0, 12}]

Out[16]= 492.232
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22) $x(t) = (1+t, t^2, t^3)$ $\quad y(s) = (\sin s, \cos s, s - \pi/2)$

$t=0,\ s=\pi/2$; $\quad x'(0) = (1,0,0)$ $\quad y'(\pi/2) = (0,-1,1)$

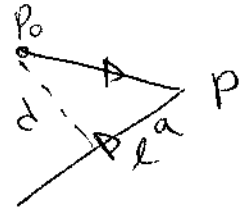
$x'(0) \cdot y'(\pi/2) = 0 \implies \boxed{\theta = \pi/2}$

1.7

30) ℓ: $\dfrac{x-1}{6} = \dfrac{y-2}{5} = \dfrac{z+3}{4}$ has direction vector

$v = (6,5,4)$ and the point $P=(1,2,-3)$ is on ℓ.

$P_0 = (8,7,9)$ so $\quad \overrightarrow{P_0P} = -(7,5,12)$



and we have $\quad a = \left| \dfrac{\overrightarrow{P_0P} \cdot v}{|v|} \right| = \dfrac{(7,5,12)\cdot(6,5,4)}{\sqrt{36+25+16}}$

$= \dfrac{115}{\sqrt{77}}$, so that $\quad d^2 = |P_0P|^2 - a^2 = 218 - \dfrac{13{,}225}{77}$

$= \dfrac{3561}{77}$ so that $\quad \boxed{d = \sqrt{\dfrac{3561}{77}}}$

6) <u>2.1</u> 6) a) $T(0,0)=15$ means the temp was 15° C at 12 CST in LA on 1/1/1996

b) The level curves are isotherms, and run east-west

c) $T(1400,1100) = -15$

d) They should be closed curves, shrinking to a single point.

12) If $z=-4$, the level curve is $x=y^2+8$
If $z=0$, the LC is $x=y^2+4$
and if $z=4$, the LC is $x=y^2$.

7) $f(x,y) = x^2 - y^3$

a) $x(t) = (t+2, 1)$ $\quad f(x(t)) = (t+2)^2 - 1$

$\frac{d}{dt} f(x(t)) = 2(t+2)$ at $t=0$ it's $\boxed{4}$

b) $x(t) = (2, t+1)$ $\quad f(x(t)) = 4 - (t+1)^3$

$\frac{d}{dt} f(x(t)) = -3(t+1)^2$ at $t=0$ it's $\boxed{-3}$

c) $x(t) = \frac{(t, 2t)}{\sqrt{5}} + (2,1) = \left(\frac{t}{\sqrt{5}}+2, \frac{2t}{\sqrt{5}}+1\right)$

$f(x(t)) = \left(\frac{t}{\sqrt{5}}+2\right)^2 - \left(\frac{2t}{\sqrt{5}}+1\right)^3$

$\frac{\partial}{\partial t} f(x(t)) = \frac{2}{\sqrt{5}}\left(\frac{t}{\sqrt{5}}+2\right) - 3\left(\frac{2t}{\sqrt{5}}+1\right)^2 \cdot \frac{2}{\sqrt{5}}$ at $t=0$ it's $4-6 =$ [illegible]

8) encore je le fait.

a) $X(t) = (t+1, 1, 1)$ $\quad \frac{\partial f(X(t))}{\partial t}\Big|_{t=0} = 2\cdot 3(t+1)\Big|_{t=0} = \boxed{6}$

b) $X(t) = (1, t+1, 1)$ $\quad \frac{\partial f(X(t))}{\partial t}\Big|_{t=0} = \boxed{5}$

c) $X(t) = (1, 1, t+1)$ $\quad \frac{\partial f(X(t))}{\partial t}\Big|_{t=0} = -2\cdot 7(t+1)\Big|_{t=0} = -14$

d) $X(t) = \frac{(3t, 4t, 12t)}{13} + (1,1,1)$

$$\frac{\partial}{\partial t} f(X(t))\Big|_{t=0} = \frac{\partial}{\partial t}\left(3\left(\frac{3t}{13}+1\right)^2 + 5\left(\frac{4t}{13}+1\right) - 7\left(\frac{12t}{13}+1\right)^2\right)\Big|_{t=0}$$

$$= \frac{1}{13}\left(18 + 20 - 168\right) = \boxed{-10}$$

9) $F = (2xy^2 + 3x^2, 2x^2y)$

a) $\gamma(t) = (t, t)$ $\quad F(\gamma(t))\cdot\gamma'(t)\,dt = (2t^3+3t^2, 2t^3)\cdot(1,1)$
$= 4t^3 + 3t^2\,dt$ so the integral is $\int_0^1 (4t^3+3t^2) = [t^4 + t^3$
$= \boxed{2}$

b) $\gamma_1(t) = (t, 0)$ $\quad \gamma_2(t) = (1, t)$

$$I = \int_0^1 3t^2\,dt + \int_0^1 (2t^2+3, 2t)\cdot(0,1)\,dt$$

$$= t^3\Big]_0^1 + t^2\Big]_0^1 = \boxed{2}$$

c) ~~Pick~~ $\gamma_1(t) = (0,t)$ $\gamma_2(t) = (t,1)$

$$I = \int_0^1 (0,0)\cdot(0,1)\,dt + \int_0^1 (2t+3t^2, 2t^2)\cdot(1,0)$$

$$= \int_0^1 (2t+3t^2)\,dt = t^2+t^3\Big]_0^1 = \boxed{2}$$

Imagin that! Quel coincidence!

10) $F = (xy, y^2-z, 3y)$

$\gamma(t) = (t,t,t)$

$$F(\gamma(t))\cdot\gamma'(t)\,dt = (t^2, t^2-t, 3t)\cdot(1,1,1)\,dt$$

$$= 2t^2+2t\,dt$$

so the integral is $\displaystyle\int_0^1 2t^2+2t\,dt = \frac{2}{3}t^3+t^2\Big]_0^1$

$$= \frac{2}{3}+1 = \boxed{\frac{5}{3}}$$