③ O-Z 2.7/26

$f = x^2 + bxy + y^2$

$\nabla f = (2x + by,\ bx + 2y)$

So $\nabla f(0,0) = 0$

$H_f = \begin{bmatrix} 2 & b \\ b & 2 \end{bmatrix} \left(= \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \right)$

$\det H_f(0,0) = 4 - b^2$

Local max/min if $|H_f| > 0$: $4 - b^2 > 0$

$4 > b^2$

$2 > |b|$

If $|b| < 2$, note that $\det H_f > 0$ and $f_{xx} > 0$.

So $\boxed{\text{Local min if } |b| < 2}$

(No local max at $(0,0)$)

Saddle point if $|H_f| < 0$: $4 - b^2 < 0$

$2 < |b|$

So $\boxed{\text{Saddle point if } |b| > 2}$

```
Show[ GraphicsArray[ {{ContourPlot[x^2 + y^2 + 4 x y , {x, -4, 4}, {y, -4, 4}],
     ContourPlot[x^2 + y^2 + 3 x y , {x, -4, 4}, {y, -4, 4}]},
    {ContourPlot[x^2 + y^2 + 1 x y , {x, -4, 4}, {y, -4, 4}],
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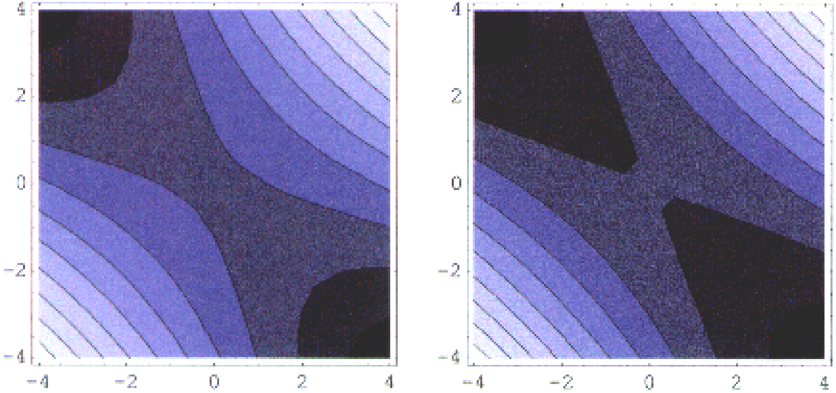
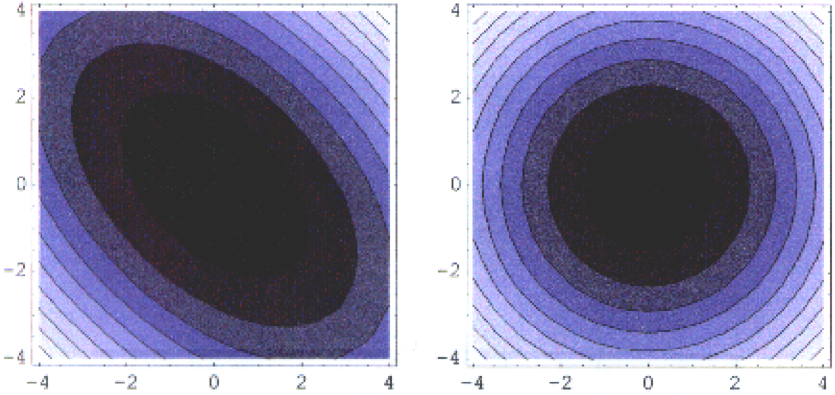


$|b| > 2$



$|b| < 2$

0-Z 2. 7/26

(3c) $\det H_f(0,0) = 0$ if $4 - b^2 = 0 \rightarrow b = \pm 2$

If $b = 2$: $f = x^2 + 2xy + y^2 = (x+y)^2$

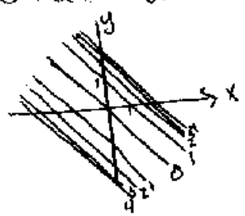
The level curves are $(x+y)^2 = c$ (note $c > 0$)

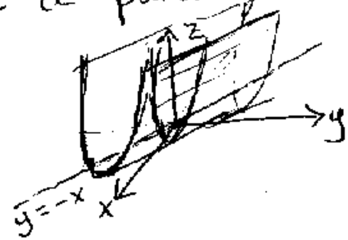
$$x + y = \pm\sqrt{c}$$

$$y = -x \pm \sqrt{c}$$

So the level curves are parallel lines to $y = -x$

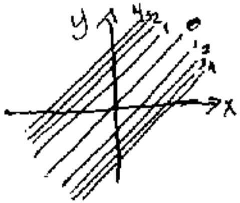
The surface would be a parabolic cylinder, with axis $y = -x$:

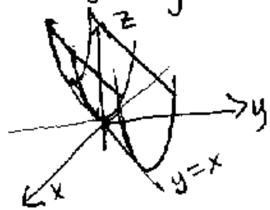




If $b = -2$, $f = x^2 - 2xy + y^2 = (x-y)^2$

Level curves would be $y = x \pm \sqrt{c}$; again, a parabolic cylinder (with axis $y = x$)



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   {ContourPlot[x^2 + y^2 - 2 x y , {x, -4, 4}, {y, -4, 4}],
    Plot3D[x^2 + y^2 - 2 x y , {x, -4, 4}, {y, -4, 4}]}}]];
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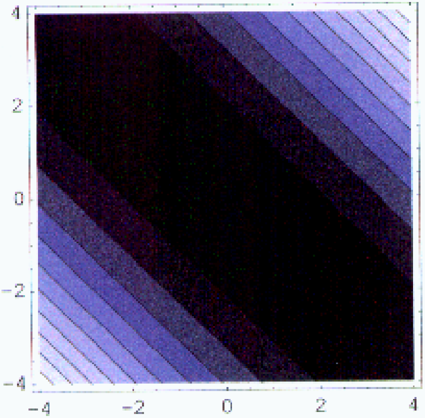
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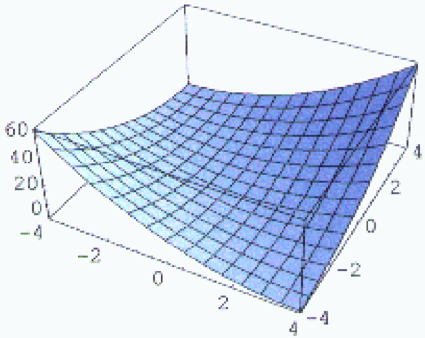
−4
−4 −2 0 2 4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40
20
0
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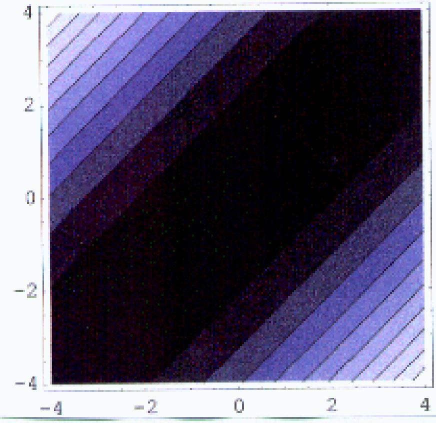
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2
0
−2
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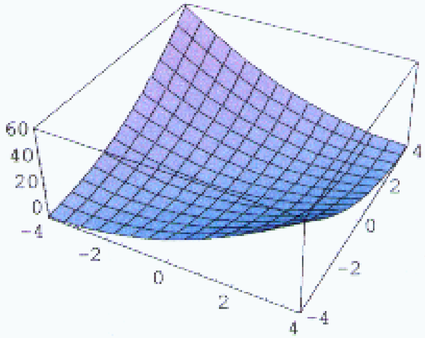
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