(4) O-Z 4.4/2:

a) Find the extrema of $f=x-y$, subject to $g=x^2+y^2-1=0$ (ie, points on the unit circle)

$\nabla f=(1,-1)$, $\nabla g=(2x,2y)$

So at the extrema, $\nabla f=\lambda\nabla g$

$$(1,-1)=\lambda(2x,2y)$$

gives us 2 equations:

$$1=\lambda(2x)$$
$$-1=\lambda(2y)$$

Can we divide out λ's? Only if lamba $\neq 0$. A quick check shows that $\lambda=0$ leads to $(x,y)=0$ which does not satisfy the constraint. Hence, $\lambda\neq 0$. Dividing

$$\frac{1}{-1}=\frac{\cancel{2\lambda}x}{\cancel{2\lambda}y}$$

$$y=-x$$

Now imposing constraint $x^2+y^2-1=0$,

$$x^2+(-x)^2-1=0$$
$$2x^2=1$$
$$x=\pm\sqrt{2}/2, \text{ and } y=-x \Rightarrow y=\mp\sqrt{2}/2$$

Hence extrema may occur at $(\sqrt{2}/2,-\sqrt{2}/2)$ and $(-\sqrt{2}/2,\sqrt{2}/2)$

$f(\sqrt{2}/2,-\sqrt{2}/2)=\sqrt{2}$ ← Max

$f(-\sqrt{2}/2,\sqrt{2}/2)=-\sqrt{2}$ ← Min

At the max $(\sqrt{2}/2,-\sqrt{2}/2)$, $\nabla f=(1,-1)$ and $\nabla g=(\sqrt{2},-\sqrt{2})$

At the min $(-\sqrt{2}/2,\sqrt{2}/2)$, $\nabla f=(1,-1)$ and $\nabla g=(-\sqrt{2},\sqrt{2})$

b) Find extrema of $f=xy$, subject to $g=x^2+y^2-1=0$

$\nabla f=(y,x)$, $\nabla g=(2x,2y)$

At extrema, $\nabla f=\lambda\nabla g$

$$(y,x)=\lambda(2x,2y)$$

Gives $y=2\lambda x$, $x=2\lambda y$

Summing $y+x=2\lambda(x+y)$

$$0=(2\lambda-1)(x+y)$$

So either $\lambda=1/2$ or $-x=y$

If $\lambda=1/2$, $y=x$.
Then $x^2+y^2-1=0\rightarrow 2x^2=1\Rightarrow x=\pm\sqrt{2}/2$, $y=\pm\sqrt{2}/2$

If $-x=y$,
then $x^2+y^2-1=0\rightarrow 2x^2=1\rightarrow x=\pm\sqrt{2}/2$, $y=\mp\sqrt{2}/2$

This gives 4 pts:

$(\sqrt{2}/2,\sqrt{2}/2)$: $f=1/2$ (Max). $\nabla f=(\sqrt{2}/2,\sqrt{2}/2)$, $\nabla g=(\sqrt{2},\sqrt{2})$

$(-\sqrt{2}/2,-\sqrt{2}/2)$: $f=1/2$ (Max) $\nabla f=(-\sqrt{2}/2,-\sqrt{2}/2)$, $\nabla g=(-\sqrt{2},-\sqrt{2})$

$(\sqrt{2}/2,-\sqrt{2}/2)$: $f=-1/2$ (Min) $\nabla f=(-\sqrt{2}/2,\sqrt{2}/2)$, $\nabla g=(-\sqrt{2},-\sqrt{2})$

$(-\sqrt{2}/2,\sqrt{2}/2)$: $f=-1/2$ (Min) $\nabla f=(\sqrt{2}/2,-\sqrt{2}/2)$, $\nabla g=(-\sqrt{2},\sqrt{2})$

4) OZ 4.4/2..

c) Find extrema of $f=x^2+y^2$ (distance from origin!) subject to $g=x+y-2=0$

$$\nabla f=(2x,2y), \quad \nabla g=(1,1)$$

At extrema, $\nabla f=\lambda \nabla g$

$$2x=\lambda(1)$$
$$2y=\lambda(1)$$

Subtracting, $2(x-y)=0 \rightsquigarrow x=y$.

Now imposing constraint $x+y-2=0$, we have $2x-2=0 \rightsquigarrow x=1, y=1$

From a geometric argument, we are finding extrema of the distance from the origin to a point on the line $y=-x+2$. Clearly, the extrema we find is a minimum (the $\perp$ distance); there is no maximum value, since as $x\to\infty$, $y\to-\infty$, and $f\to\infty$.

$$\boxed{\nabla f(1,1)=(2,2), \quad \nabla g(1,1)=(1,1)}$$

d) Find extrema of $f=2x+y+z$, subject to $g=x^2+y^2+z^2-6=0$
(i.e, points lie on a sphere of radius $\sqrt{6}$)

$$\nabla f=(2,1,1), \quad \nabla g=(2x,2y,2z).$$

At extrema, $\nabla f=\lambda\nabla g$

$$2=\lambda(2x)$$
$$1=\lambda(2y)$$
$$1=\lambda(2z)$$

If $\lambda=0$, then this reads $2=0, 1=0, 1=0$. Hence $\lambda\neq 0$ so we can divide it out:

$$\frac{2}{1}=\frac{2\lambda x}{2\lambda y} \rightsquigarrow 2y=x$$

$$\frac{1}{1}=\frac{2\lambda y}{2\lambda z} \rightsquigarrow z=y$$

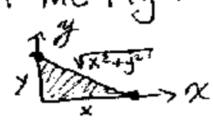
The constraint $x^2+y^2+z^2-6=0 \to (2y)^2+y^2+y^2-6=0 \rightsquigarrow 6y^2=6 \to y=\pm 1$, $x=\pm 2$, $z=\pm 1$

Possible extrema are thus $(2,1,1)$ and $(-2,-1,-1)$

$$\boxed{\begin{array}{lll} f(2,1,1)=6 \ (\text{Max}) & \nabla f=(2,1,1), & \nabla g=(4,2,2) \\ f(-2,-1,-1)=-6 \ (\text{Min}) & \nabla f=(2,1,1), & \nabla g=(-4,-2,-2) \end{array}}$$

4) OZ 4.4/8:

Set up a coordinate system with the two axes being sides of the right triangle.



Then area $A=\frac{1}{2}xy$, and perimeter $P=x+y+\sqrt{x^2+y^2}$.

Maximize $A=\frac{1}{2}xy$ subject to $P=x+y+\sqrt{x^2+y^2}=100$

$$\nabla A=\left(\tfrac{1}{2}y, \tfrac{1}{2}x\right). \quad \nabla P=\left(1+\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}, \ 1+\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}\right)$$

At extrema, $\nabla A=\lambda\nabla P$.

So $\frac{1}{2}y=\lambda\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)$.

$$\frac{1}{2}x=\lambda\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)$$

If $\lambda=0$, $y=0$, $x=0$. This doesn't satisfy contraint hence $\lambda\neq 0$.

So $\frac{y}{2\lambda}=1+\frac{x}{\sqrt{x^2+y^2}}$, $\quad \frac{x}{2\lambda}=1+\frac{y}{\sqrt{x^2+y^2}}$

4) 4.4/8 cont

Let's try to get rid of the square roots, by equating them.

$$y\left(\frac{1}{2\lambda}y-1\right)=y\left(\frac{x}{\sqrt{x^2+y^2}}\right) \qquad x\left(\frac{1}{2\lambda}x-1\right)=x\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

So $\frac{1}{2\lambda}y^2-y=\frac{1}{2\lambda}x^2-x$

$$0=\frac{1}{2\lambda}(x^2-y^2)-(x-y)$$

$$0=\left[\frac{1}{2\lambda}(x+y)-1\right](x-y)$$

So either $\frac{1}{2\lambda}(x+y)=1$, or $x=y$

If $\frac{1}{2\lambda}(x+y)=1$, then $y=2\lambda-x$

Let's plug this back into the Lagrange eqn's:

$$\frac{2\lambda-x}{2}=\lambda\left(1+\frac{x}{\sqrt{x^2+(2\lambda-x)^2}}\right)$$

$$1-\frac{x}{2\lambda}=1+\frac{x}{\sqrt{x^2+4\lambda^2-4\lambda x+x^2}}$$

So $-2\lambda=\sqrt{2x^2-4\lambda x+4\lambda^2}$

Squaring, $4\lambda^2=2x^2-4\lambda x+4\lambda^2$

$$0=2x(x-2\lambda)$$

So $x=0 \Rightarrow y=2\lambda$ OR $x=2\lambda \Rightarrow y=0$

If $x+y+\sqrt{x^2+y^2}=100$, then

$(0,2\lambda) \rightsquigarrow 2\lambda+2\lambda=100$, $\lambda=25$ $\qquad$ $(2\lambda,0)\rightsquigarrow 2\lambda+2\lambda=100$, $\lambda=25$

$(0,50)$ or $(50,0)$ are possible extremas. For both, $A=\frac{1}{2}xy=0$, and since x & y are in the first quadrant, these are MINIMA.

So let's now try $x=y$ (we already tried $\frac{1}{2\lambda}(x+y)=1$):

If $x=y$, then $x+y+\sqrt{x^2+y^2}=100$

$$x+x+\sqrt{x^2+x^2}=100$$

$$x(2+\sqrt{2})=100$$

$$x=\frac{100}{2+\sqrt{2}}=y \quad (\approx 29.3)$$

$$A\left(\frac{100}{2+\sqrt{2}},\frac{100}{2+\sqrt{2}}\right)=\frac{1}{2}\frac{100^2}{(2+\sqrt{2})^2}=\frac{5000}{4+4\sqrt{2}+2}=\frac{2500}{3+2\sqrt{2}}\left(\frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right)=\frac{2500(3-2\sqrt{2})}{9-8}=2500(3-2\sqrt{2}) \approx 428.9$$

So the maximum occurs when the fencing forms an isosceles right triangle, with $\boxed{x=y=\frac{100}{2+\sqrt{2}}\approx 29.3}$ and $\boxed{A\approx 428.9}$ (measured in feet).

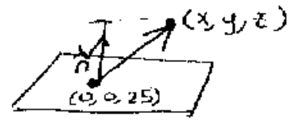
(5) Lagrange multiplier worksheet!

[8] There are a number of ways to solve this problem. Let's combine some of our old vector techniques with our new Lagrange techniques.

First, what is the distance of any ol' point to a plane $2x+3y+z=25$?

Recall, we need a point on the plane; $(0,0,25)$ will do quite well.

Construct a difference vector from the point on the plane $((0,0,25))$ to our point $((x,y,z))$:

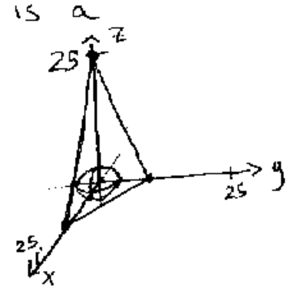


$$\vec{v} = (x,\ y,\ z-25).$$

To find perpendicular distance to the plane we simply want the projection of this vector onto the normal to the plane. We read the components of the normal vector off of the scalar eqn: $\vec{n} = (2,3,1)$.

$$\text{So } d = \left|\mathrm{Proj}_{\vec{n}}\vec{v}\right| = \left|\frac{\vec{n}\cdot\vec{v}}{\|\vec{n}\|}\right| = \left|\frac{2x+3y+(z-25)}{\sqrt{4+9+1}}\right| = \left|\frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y + \frac{1}{\sqrt{14}}z - \frac{25}{\sqrt{14}}\right|$$

Now, we'd like to find (x,y,z) on the ellipsoid such that d is a minimum. A quick sketch should show our desired point is in the first quadrant; also let's drop the absolute value bars.* Our problem is now:



Minimize: $d = \frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y + \frac{1}{\sqrt{14}}z - \frac{25}{\sqrt{14}}$,

subject to $g = x^2/4 + y^2/9 + z^2/4 = 1$.

$$\nabla d = \left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right), \quad \nabla g = \left(x/2,\ 2y/9,\ z/2\right)$$

*Note — If $\mathrm{Proj}_{\vec{n}}\vec{v} < 0$, then $\nabla d \rightsquigarrow -\nabla d$. This $\ominus$ can be absorbed by the constant λ.

At extrema, $\nabla d = \lambda \nabla g$,

$$\text{so } \frac{2}{\sqrt{14}} = \frac{\lambda}{2}x, \quad \frac{3}{\sqrt{14}} = \frac{2\lambda}{9}y, \quad \frac{1}{\sqrt{14}} = \frac{\lambda}{2}z$$

If $\lambda = 0$, $\frac{2}{\sqrt{14}} = 0$, so $\lambda \neq 0$. Dividing,

$$\frac{2/\sqrt{14}}{3/\sqrt{14}} = \frac{\lambda/2\, x}{2\lambda/9\, y} \rightsquigarrow \frac{8}{27}y = x \rightsquigarrow y = \frac{27}{8}x$$

$$\frac{2/\sqrt{14}}{1/\sqrt{14}} = \frac{\lambda/2\, x}{\lambda/2\, z} \rightsquigarrow 2z = x \rightsquigarrow z = \frac{1}{2}x$$

Now using constraint, $x^2/4 + y^2/9 + z^2/4 = 1$, $\quad \frac{x^2}{4} + \frac{(27/8)^2x^2}{9} + \frac{(1/2)^2x^2}{4} = 1$

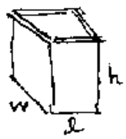
$$x^2\left(\frac{1}{4} + \frac{81}{64} + \frac{1}{16}\right) = 1$$

$$\left(\frac{101}{64}\right)x^2 = 1 \rightsquigarrow x = \pm\frac{8}{\sqrt{101}}.$$

From previous arguments, minimum occurs at the positive root. So our desired point is $\boxed{\left(\frac{8}{\sqrt{101}}, \frac{27}{\sqrt{101}}, \frac{4}{\sqrt{101}}\right)}$. Minimum distance is $\approx |-3.996| \approx 4$

⑤ Lagrange Wksheet:

$\boxed{10}$ First, let's make a rectangular box:



$V = wlh$, Surface area $g = 2wh + 2hl + wl = 36$

$\nabla V = (lh, wh, wl)$, $\nabla g = (2h+l, 2h+w, 2w+2l)$

Extrema occur at $\nabla V = \lambda \nabla g$

So $lh = \lambda(2h+l)$, ①
$wh = \lambda(2h+w)$, ②
$wl = \lambda(2w+2l)$ ③

If $\lambda = 0$, then at least 2 of the w, l, h dimensions are 0, hence $V=0$.
This is clearly not the box of greatest volume, so lets discard $\lambda = 0$.

① $(l-2\lambda)h = l\lambda$ Can divide by $l-2\lambda$ unless $l-2\lambda=0$. In this case, $l\lambda=0$, which we reject by similar arguments as before

$$h = \frac{l\lambda}{l-2\lambda}$$

② $(w-2\lambda)h = w\lambda$

$$h = \frac{w\lambda}{w-2\lambda}$$

③ $wl = 2\lambda(w+l) \rightarrow w = \frac{2\lambda l}{l-2\lambda} = 2h$

$\rightarrow l = \frac{2\lambda w}{w-2\lambda} = 2h$

So $w = 2h$, $l = 2h$. Imposing constraint, $2wh + 2hl + wl = 36$

$$2(2h)h + 2h(2h) + (2h)(2h) = 36$$
$$3 \cdot 4h^2 = 36$$
$$h^2 = 3 \rightsquigarrow h = \pm\sqrt{3},\quad w = l = \pm 2\sqrt{3}$$

Taking positive roots,

$\boxed{V_{max} = (\sqrt{3})(2\sqrt{3})(2\sqrt{3}) = 12\sqrt{3}} \approx 20.78$

Now, a cylindrical box:

$V = \pi r^2 h$, Surface area $g = \pi r^2 + 2\pi r h = 36$

$\nabla V = (2\pi r h, \pi r^2)$, $\nabla g = (2\pi r + 2\pi h, 2\pi r)$

At extrema, $\nabla V = \lambda \nabla g$
$2\pi r h = \lambda(2\pi r + 2\pi h)$
$\pi r^2 = \lambda(2\pi r)$

If $r=0$, then clearly volume is not a maximum. So, dividing, $r = 2\lambda$.
Substituting, $2\lambda h = \lambda(2\lambda + h) \rightsquigarrow (2\lambda - \lambda)h = 2\lambda^2$. either $\lambda = 0$ ($\rightarrow r=0$, reject)! or $h = 2\lambda$.
So $r=h$. From constraint, $\pi r^2 + 2\pi r^2 = 36 \rightsquigarrow r^2 = 12/\pi$, $r = \pm\sqrt{12/\pi} = h$

Taking positive roots,

$\boxed{V_{max} = \pi\left(\frac{12}{\pi}\right)\sqrt{\frac{12}{\pi}} = 12\sqrt{3}\left(\frac{2}{\sqrt{\pi}}\right)} \approx 23.45$

Since $\pi < 4$, the cylindrical box has a greater capacity, by a factor of $2/\sqrt{\pi}$ (an amount difference of $12\sqrt{3}\left(\frac{2}{\sqrt{\pi}} - 1\right) \approx 2.67$)

5 Lagrange Multipliers Wksht,

12 ⓐ Minimize $B = 35x + 16y$ subject to $P = 500x^{.7}y^{.5} = 40000$

$\nabla B = (35, 16)$ $\qquad$ $\nabla P = ((.7)(500)x^{-.3}y^{.5}, .5(500)x^{.7}y^{-.5})$

Extrema occur at $\nabla B = \lambda \nabla P$

$$35 = \lambda(350x^{-.3}y^{.5})$$
$$16 = \lambda(250x^{.7}y^{-.5})$$

Convince yourself that λ, x, y are not 0. (If any are, then $35 = 0$!!)

$$\frac{35}{16} = \frac{\lambda \cdot 350x^{-.3}y^{.5}}{\lambda \cdot 250x^{.7}y^{-.5}} = \frac{7}{5}\frac{y}{x} \rightsquigarrow \frac{25}{16}x = y$$

Now impose the constraint:

$$500x^{.7}\left(\frac{25}{16}x\right)^{.5} = 40000$$
$$x^{1.2} = 80\left(\frac{25}{16}\right)^{-.5}$$
$$\boxed{x = 32, \quad y = 50}$$

This corresponds to a cost of $\boxed{B = 35(32) + 16(50) = \$1920}$

ⓑ Minimize $P = 500x^{.7}y^{.5}$ subject to $B = 35x + 16y = 4800$

$\nabla P = (350x^{-.3}y^{.5}, 250x^{.7}y^{-.5})$, $\quad \nabla B = (35, 16)$ as before.

Extrema at $\nabla P = \lambda \nabla B$, as before

$$350x^{-.3}y^{.5} = \lambda(35)$$
$$250x^{.7}y^{-.5} = \lambda(16)$$

Dividing, $\frac{7}{5}\frac{y}{x} = \frac{35}{16} \rightsquigarrow y = \frac{25}{16}x$, as before! This will always be true under optimal conditions.

Impose the constraint:

$$35x + 16\left(\frac{25}{16}x\right) = 4800$$
$$60x = 4800$$
$$\boxed{x = 80, \quad y = 125}$$

This corresponds to a lollipop production of $\boxed{P = 500(80)^{.7}(125)^{.5} \approx 120{,}112}$

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13 a) At $t = 10$, $x(t) = 20 + 80 = 100$
$y(t) = 45 + 80 = 125$

$$\boxed{\begin{aligned} B(t=10) &= 100x(t) + 120y(t)\big|_{t=10} \\ &= 100(100) + 120(125) = \$25000 \end{aligned}}$$

$$\boxed{\begin{aligned} P(t=10) &= 300x(t)^{1/2}y(t)^{1/3}\big|_{t=10} \\ &= 300(100)^{1/2}(125)^{1/3} = 15000 \end{aligned}}$$

⑤ Lagrange, cont.

ⓑ $\frac{\partial P}{\partial B} = \frac{\partial P}{\partial x}\frac{dx}{\partial B} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial B}$

If we only increase Budget on labor, $\partial y/\partial B = 0$

$\frac{\partial x}{\partial B} = \frac{1}{\partial B/\partial x}$. $\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}(100x + 120y) = 100$

And $\frac{\partial P}{\partial x} = .5(300x^{-.5}y^{1/3}) = 150x^{-1/2}y^{1/3}$

So $\left.\frac{\partial P}{\partial B}\right|_{\$\to labor} = (150x^{-1/2}y^{1/3})\left(\frac{1}{100}\right)$

At the current point, $\boxed{\left.\frac{\partial P}{\partial B}\right|_{\$\to labor}(100,125) = 1.5(100^{-1/2} \cdot 125^{1/3}) = .75} \approx \frac{\Delta P}{\Delta B}$

Since ΔB is so small, this instantaneous rate of change should be $\approx \Delta P/\Delta B$. Let's calculate ΔP!

$1 ⇝ budget for x means x increases by $\frac{1}{100}$.

So new $(x,y) = (100.01, 125)$.

$P(100.01, 125) = 300(100.01)^{1/2}(125)^{1/3} \approx 15000.74998$

$\Delta P = 15000.74998 - 15000 = .74998$

And $\Delta B = \$1$

So $\boxed{\frac{\Delta P}{\Delta B} \approx \frac{.74998}{1} = .74998}$

ⓒ Similarly, $\frac{\partial P}{\partial B} = \frac{\partial P}{\partial y}\frac{\partial y}{\partial B}$, if $\partial x/\partial B = 0$ (all Budget increase → capital).

$\frac{\partial B}{\partial y} = 120$, So $\frac{\partial y}{\partial B} = \frac{1}{120}$

$\frac{\partial P}{\partial y} = \frac{1}{3}(300x^{1/2}y^{-2/3}) = 100x^{1/2}y^{-2/3}$

$\left.\frac{\partial P}{\partial B}\right|_{\$\to cap} = (100x^{1/2}y^{-2/3})\left(\frac{1}{120}\right)$

At current point, $\boxed{\left.\frac{\partial P}{\partial B}\right|_{\$\to cap} = \frac{100}{120}100^{1/2} \cdot 125^{-2/3} = \frac{1}{3}} \approx .33333 \approx \frac{\Delta P}{\Delta B}$

Or, calculating ΔP, $1 budget increase, increases y by $\frac{1}{120}$.

new $x,y = (100, 125 + \frac{1}{120}) = (100, 125.00833)$

$P(100, 125.00833) = 300(100)^{1/2}(125.00833)^{1/3} \approx 15000.33333$

$\Delta P \approx .33333$, $\Delta B = \$1$

So $\boxed{\frac{\Delta P}{\Delta B} = \frac{.33333}{1} = .33333}$

(5) Lagrange multipliers wksheet,
[13] cont

(d) Hmm... getting a maximal production for a fixed budget... sounds like Lagrange multipliers to me!

Maximize $P = 300x^{1/2}y^{1/3}$ given $B = 100x + 125y = 25000$

$\nabla P = (150x^{-1/2}y^{1/3}, 100x^{1/2}y^{-2/3}) \qquad \nabla B = (100, 125)$

$\nabla P = \lambda \nabla B$

$150x^{-1/2}y^{1/3} = \lambda(100)$

$100x^{1/2}y^{-2/3} = \lambda(125)$

$\frac{150}{100}\frac{y}{x} = \frac{100}{125} \rightsquigarrow y = \frac{8}{15}x$

Using constraint, $100x + 125\left(\frac{8}{15}x\right) = 25000$

$\frac{500}{3}x = 25000$

$\boxed{x = 150, \quad y = \frac{8}{15}(150) = 80}$

with $x=150$, $y=80$, production is $\boxed{P = 300(150)^{1/2}(80)^{1/3} = 15831.79702}$

(e) The optimal resource allocation always has $y = 8/15\, x$, since we derived that independently from the budget constraint. So with our new budget $= 25001$, $100x + 125\left(\frac{8}{15}x\right) = 25001$

$\frac{500}{3}x = 25001$

$x = 150.006, \quad y = 80.0032$

$P = 300(150.006)^{1/2}(80.0032)^{1/3} \approx 15832.32475$

$\Delta P = 15832.32475 - 15831.79702 \approx .52772$

$\boxed{\left.\frac{\Delta P}{\Delta B}\right|_{optimal} = .52772}$

Quirkily enough, $\left.\frac{\partial P}{\partial B}\right|_{optimal}$ is actually our Lagrange multiplier λ.

Why? $\frac{dP}{dB} = \frac{\partial P}{\partial x}\frac{dx}{dB} + \frac{\partial P}{\partial y}\frac{dy}{dB}$. But Lagrange multipliers tells us that optimally, $\frac{\partial P}{\partial x} = \lambda\frac{\partial B}{\partial x}$, $\frac{\partial P}{\partial y} = \lambda\frac{\partial B}{\partial y}$

So $\left.\frac{dP}{dB}\right|_{optimal} = \lambda\left(\frac{\partial B}{\partial x}\frac{dx}{dB} + \frac{\partial B}{\partial y}\frac{dy}{dB}\right) = \lambda\left(\frac{dB}{dB}\right) = \lambda$

What is λ? Well, $150x^{-1/2}y^{1/3} = \lambda(100)$. Using $x=150$, $y=80$,

$\lambda = 1.5(150)^{-1/2}(80)^{1/3} = .52773$

So $\boxed{\lambda = \left.\frac{\partial P}{\partial B}\right|_{optimal} = .52773 \approx \left.\frac{\Delta P}{\Delta B}\right|_{optimal}}$