⑥ a) On the surface of the moon, we have a Lagrange Mult. problem

Maximize $T = 50(1-x^2-y^2-z^2) + 10(\sqrt{3}x + z)$ subj to $g = x^2+y^2+z^2 = 1$

$\nabla T = (-100x + 10\sqrt{3}, -100y, -100z+10)$ $\qquad \nabla g = (2x, 2y, 2z)$

At extrema, $\nabla T = \lambda \nabla g$

$-100x + 10\sqrt{3} = \lambda 2x$

$-100y = \lambda 2y \longrightarrow 0 = y(\lambda+50)$, so $y=0$ or $\lambda = -50$

$-100z + 10 = \lambda 2z$

If $\lambda = -50$, $-100x + 10\sqrt{3} = (-50)2x \rightsquigarrow 10\sqrt{3} = 0$. So $\lambda \neq -50$.

So, $\boxed{y=0}$ $\quad x, z$ cannot $=0$, since that would lead to $10\sqrt{3}, 10 = 0$.

So we can divide to eliminate λ:

$$-50 + \frac{5\sqrt{3}}{x} = \lambda = -50 + \frac{10/2}{z}$$

$$\frac{5\sqrt{3}}{x} = \frac{5}{z}$$

$$\boxed{z = x/\sqrt{3}}$$

Apply constraint $x^2+y^2+z^2=1 \to x^2 + 0 + \frac{x^2}{3} = 1 \rightsquigarrow \frac{4}{3}x^2 = 1 \rightsquigarrow x = \pm\frac{\sqrt{3}}{2}$

$z = \pm \frac{1}{2}$

Possible extrema: $(\sqrt{3}/2, 0, 1/2)$, $(-\sqrt{3}/2, 0, -1/2)$

$T(\sqrt{3}/2, 0, 1/2) = 50(1 - \frac{3}{4} - 0 - \frac{1}{4}) + 10(\frac{1}{2}+\frac{1}{2}) = 10$

$T(-\sqrt{3}/2, 0, -1/2) = 50(1 - \frac{3}{4} - 0 - \frac{1}{4}) + 10(-\frac{1}{2}-\frac{1}{2}) = -10$

So $\boxed{\text{hottest point on surface of moon is at } (\sqrt{3}/2, 0, 1/2)}$

b) Having already considered $\nabla T = \lambda \nabla g$ (extrema on surface), we should now check $\nabla g = 0$: $(x,y,z) = (0,0,0)$

And of course, the unconstrained problem, $\nabla T = 0$

$(-100x + 10\sqrt{3}, -100y, -100z+10) = (0,0,0)$

$-100x + 10\sqrt{3} = 0 \qquad -100y = 0 \qquad -100z + 10 = 0$

$\frac{\sqrt{3}}{10} = x \qquad y = 0 \qquad \frac{1}{10} = z$

Is $(\sqrt{3}/10, 0, 1/10)$ inside sphere? $x^2+y^2+z^2 = \frac{3}{100} + 0 + \frac{1}{100} = \frac{1}{25} < 1$, yes!

$T(0,0,0) = 0$

$T(\frac{\sqrt{3}}{10}, 0, \frac{1}{10}) = 50(1 - \frac{3}{100} - 0 - \frac{1}{100}) + 10(\frac{3}{10} + \frac{1}{10}) = 52$

So the temperature is $\boxed{\text{greatest at } (\sqrt{3}/10, 0, 1/10)}$

c) We have the same candidate points; we simply take the one giving smallest T.

So the temperature is $\boxed{\text{least at } (-\sqrt{3}/2, 0, -1/2)}$

7 Let's do this problem 2 different ways.

Way 1: Vectors & Calc 1 Material

The planes $2x+y+z=2$ & $x-y-3z=4$ intersect in a line that is $\perp$ to both of their normal vectors $(2,1,1)$ and $(1,-1,-3)$. We can find such a vector using the cross product:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -1 & -3 \end{vmatrix} = (-2, 7, -3) \text{ is in direction of line}$$

Now we just need a point on this line — a point on both planes!
Choose $z=0$. Then $2x+y=2$ & $x-y=4$. Summing, $3x=6 \to x=2$
Hence $y=-2$

So $(2,-2,0)$ is on line.
Now I can describe the line parametrically:

$$\left.\begin{aligned} x &= -2t+2 \\ y &= 7t-2 \\ z &= -3t \end{aligned}\right\} t \in \mathbb{R}$$

Now, we'd like to minimize distance from this line to the origin (or distance squared). $D^2 = x^2+y^2+z^2$, for a general point.
But a point on this line must satisfy our parametric equation for some t,
So $f = D^2 = (-2t+2)^2 + (7t-2)^2 + (-3t)^2$

$$f(t) = D^2 = 4t^2 - 8t + 4 + 49t^2 + -28t + 4 + 9t^2$$
$$= 62t^2 - 36t + 8$$

Since f is a function of t only, I can minimize it using calc 1 techniques!

$$f'(t) = 124t - 36 = 0$$
$$124t = 36$$
$$t = 9/31$$

Hence $x(t) = -\frac{18}{31} + 2 = \frac{44}{31}$, $y(t) = \frac{63}{31} - 2 = 1/31$, $z(t) = -27/31$

So the closest point on the intersection of those planes is $\boxed{\left(\frac{44}{31}, \frac{1}{31}, -\frac{27}{31}\right)}$

Way 2: Lagrange Multipliers with Multiple Constraints

Minimize $f = x^2+y^2+z^2$ subject to $g = 2x+y+z=2$ AND $h = x-y-3z=4$

Key to L.M. problems with multiple constraints:

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$$ ← Think about why this must be!

$\nabla f = (2x, 2y, 2z)$. $\nabla g = (2,1,1)$. $\nabla h = (1,-1,-3)$

$$2x = 2\lambda_1 + \lambda_2$$
$$2y = \lambda_1 - \lambda_2$$
$$2z = \lambda_1 - 3\lambda_2$$

Let's eliminate λ_2:

Adding 1st & 2nd: $2x + 2y = 2\lambda_1 + \lambda_1 \rightarrow 2(x+y) = 3\lambda_1$

Adding 3* 1st & 3rd $6x + 2z = 6\lambda_1 + \lambda_1 \rightarrow 2(3x+z) = 7\lambda_1$

Eliminating λ_1:

$$\frac{2(x+y)}{3} = \lambda_1 = \frac{2(3x+z)}{7}$$

$$\left(\frac{7}{3} - 3\right)x + \frac{7}{3}y = z$$

"This doesn't look like a solution!" you cry. But don't forget, we still have two whol[e] equations we haven't used — the constraints.

g: $2x + y + z = 2$

$$2x + y + \left(-\frac{2}{3}x + \frac{7y}{3}\right) = 2$$

$$4x + 10y = 6$$

h: $x - y - 3z = 4$

$$x - y - 3\left(-\frac{2}{3}x + \frac{7y}{3}\right) = 4$$

$$3x - 8y = 4$$

$-4/3$ * this eqn + last eqn:

$$\begin{array}{r} 4x + 10y = 6 \\ -4x + \frac{32}{3}y = -\frac{16}{3} \\ \hline \frac{62}{3}y = \frac{2}{3} \end{array}$$

$$y = 1/31$$

So $x = \frac{6 - 10/31}{4} = 44/31$, $z = -\frac{2}{3}\left(\frac{44}{31}\right) + \frac{7}{3}\left(\frac{1}{31}\right) = -\frac{27}{31}$

So the closest point to origin on BOTH planes simultaneously (& thus on that line) is $\boxed{(44/31, 1/31, -27/31)}$. Same as before!

Choose whichever method you're comfortable with.

(8) First, let's find the potential mins & max's of the unbounded problem:

$f = x^2 - xy + 2y^2 - 6x - 4y$

$\nabla f = (2x - y - 6, -x + 4y - 4) = 0$ for unconstrained max/min

$2x - y - 6 = 0, \quad -x + 4y - 4 = 0$

Twice the 2^{nd} + the 1^{st} $\to$ $7y - 14 = 0$

$y = 2 \rightsquigarrow x = 4$

So $\underline{(4,2)}$ is a potential max/min.

Check out the Hessian matrix,

$H_f = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$ $\det H_f = 8 - 1 > 0$ so max or min. $f_{xx} > 0$ so local min! at $(4,2)$

Now, let's deal with the boundaries. We'll take the 3 sides one at a time.

Two ways:

Left: constraint $g = x = 0$

Extrema at $\nabla f = \lambda \nabla g$

So $2x - y - 6 = \lambda(1)$

$-x + 4y - 4 = \lambda(0)$

So $4y = x + 4$

using constraint $x = 0 \to y = 1$

Extrema on left edge is @ $\underline{(0,1)}$

$\leftarrow$ Lagrange

Calc 1 $\to$

Constraint $x = 0$ means

$f = 0 - 0y + 2y^2 - 0 - 4y$

$f'(y) = 4y - 4$

Max when $f'(y) = 0 \rightsquigarrow y = 1$

Bottom: Constraint $h = y = 0$

Extrema at $\nabla f = \lambda \nabla h$

$2x - y - 6 = \lambda(0)$

$-x + 4y - 4 = \lambda(1)$

So $x = \frac{1}{2}(y + 6)$

using constraint, $y = 0 \rightsquigarrow x = 3$

$\leftarrow$ Lagrange

Calc 1 $\to$

Constraint $y = 0$ means

$f = x^2 - 0x + 0 - 6x - 0$

$f'(x) = 2x - 6 = 0$

Max at $x = 3, \ y = 0$

Extrema @ $\underline{(3,0)}$

Diagonal Constraint $j = x + 2y = 24$

Extrema at $\nabla f = \lambda \nabla j$

$2x - y - 6 = \lambda(1)$

$-x + 4y - 4 = \lambda(2)$

$2\,eq1 + -eq2$:

$5x - 6y - 8 = 0$

$\frac{5x - 8}{6} = y$

Constraint $x + 2y = 24$

$x + \frac{5x - 8}{3} = 24$

$\frac{8x}{3} = \frac{80}{3} \rightsquigarrow x = 10, \ y = 7$

$\leftarrow$ Lagrange

Calc 1 $\to$

If $x + 2y = 24$, then $x = 24 - 2y$

$f = x^2 - xy + 2y^2 - 6x - 4y$

$f = (24 - 2y)^2 - (24 - 2y)y + 2y^2 - 6(24 - 2y) - 4y$

$f = 432 - 112y + 8y^2$

$f'(y) = -112 + 16y = 0$

$16y = 112$

$y = 7$

$x = 24 - 2y = 10$

Extrema @ $\underline{(10,7)}$

So now we have candidates for max/min's. We should also consider places where $\nabla g = 0$ (there are none) and any places where the function isn't smooth, ie, CORNERS

Our candidates are thus $(4,2)$, $(0,1)$, $(3,0)$, $(10,7)$, $(0,0)$, $(0,12)$, and $(24,0)$.

Evaluating,

$f(4,2) = -16 \longleftarrow$ Absolute minimum!

$f(0,1) = -2$

$f(3,0) = -9$

$f(10,7) = 40$

$f(0,0) = 0$

$f(0,12) = 240$

$f(24,0) = 432 \longleftarrow$ Absolute maximum!

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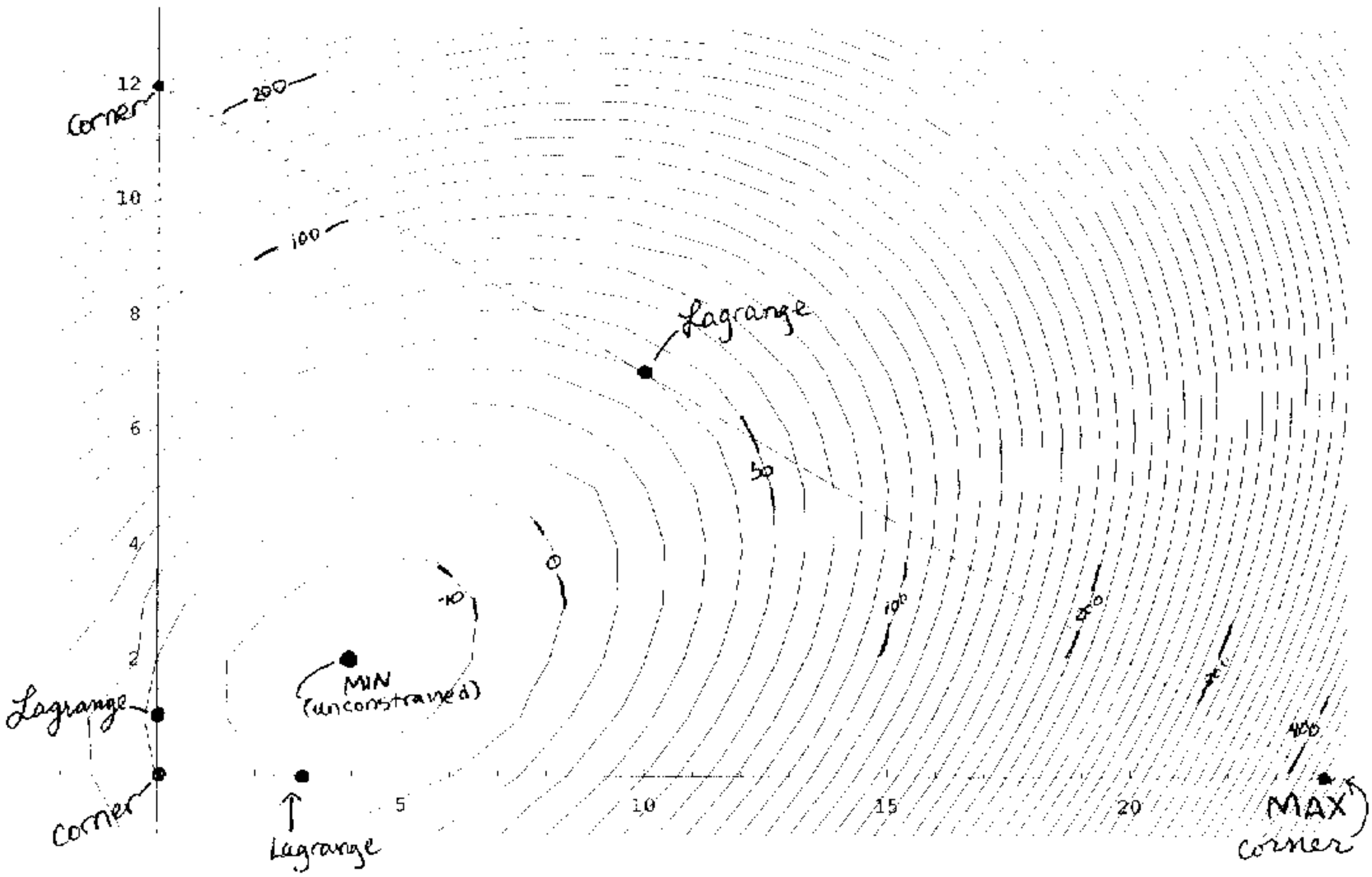
In[108]:= Show[Plot[12 - .5 x, {x, 0, 24}],
  ContourPlot[x^2 - x y + 2 y^2 - 6 x - 4 y, {x, -2, 25}, {y, -1, 13},
   ContourShading -> False, Contours -> {-10, 0, 10, 20, 30, 40, 50, 60,
     70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200,
     210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 320, 330,
     340, 350, 360, 370, 380, 390, 400, 410, 420, 430, 440, 450}]];
```



(9) First, let's find solutions to the unconstrained max/min problem

$\nabla T = (4x, z, y-4)$

Extrema at $\nabla T = 0 \rightsquigarrow$ $4x=0$, $z=0$, $y-4=0$

So $(0,4,0)$ is a possible extrema

Notice this point actually on the ellipsoid: $0^2/4 + 4^2/16 + 0^2/4 = 1$

So let's hope it shows up on our constrained problem!

Now, the constrained problem, subj to $g = x^2/4 + y^2/16 + z^2/4 = 1$

$\nabla g = (x/2,\ y/8,\ z/2)$

At extrema, $\nabla T = \lambda \nabla g$

$4x = \lambda/2\, x \longrightarrow x(4-\lambda/2) = 0$ So either $x=0$ or $\lambda/8 = 1$

$z = \lambda/8\, y$

$y - 4 = \lambda/2\, z$

$\boxed{\text{If}}$ $\lambda = 8$, then $z = y$

so $y - 4 = 4z = 4y$

$-4/3 = y = z$

And using constraint, $\frac{x^2}{4} + \frac{(-4/3)^2}{16} + \frac{(-4/3)^2}{4} = 1$

$x^2 = 16/9 \rightarrow x = \pm 4/3$

So candidates for max/min are $(4/3, -4/3, -4/3)$ and $(-4/3, -4/3, -4/3)$

(If) $x = 0$,

then $z = \lambda/8\, y$. Note if $y=0$, then $z=0$, so $0-4=0$ false! So $y \neq 0$, and we can divide

$\frac{4z}{y} = \lambda/2$

So $y - 4 = \left(\frac{4z}{y}\right) z \rightsquigarrow \frac{y^2-4y}{4} = z^2$

Now applying constraint,

$\frac{0^2}{4} + \frac{y^2}{16} + \frac{\left(\frac{1}{4}y^2 - 4y\right)}{4} = 1$

$\frac{1}{8}y^2 - \frac{1}{4}y - 1 = 0 \rightsquigarrow y^2 - 2y - 8 = 0$

$y = \frac{2 \pm \sqrt{4+32}}{2} = \frac{2 \pm 6}{2} = 4, -2$

If $y = 4$, then $z^2 = \frac{4^2 - 4\cdot 4}{4} = 0$. If $y = -2$, $z^2 = \frac{(-2)^2 - 4\cdot(\ )}{4} = 3$

$z = \pm\sqrt{3}$

So candidates are $(0,4,0)$ (yay!), and $(0,-2,\sqrt{3})$ and $(0,-2,-\sqrt{3})$

Finally consider corners (none) or points where $\nabla g = 0$: $(x,y,z) = (0,0,0)$

Extrema:

$T(0,4,0) = 150$

$T(4/3,-4/3,-4/3) \approx 160.67$ — absolute max temperatures at $(\pm 4/3, -4/3, -4/3)$

$T(-4/3,-4/3,-4/3) \approx 160.67$

$T(0,-2,\sqrt{3}) \approx 153.46$

$T(0,-2,-\sqrt{3}) \approx 146.53$ — min temperature at $(0,-2,-\sqrt{3})$

$T(0,0,0) = 150$