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Math 21a

Problem Set #5
Solution Set

1 Define $F(x,y) = 7x^2y^6 - 5xy^5 + 3y^3$

Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{14xy^6 - 5y^5}{42x^2y^5 - 25xy^4 + 9y^2} = -\dfrac{14xy^4 - 5y^3}{42x^2y^3 - 25xy^2 + 9}$

The tangent line can be calculated with the formula from 1-variable calc

$$y - y_0 = m(x - x_0)$$

where m = slope of tangent line at $(x_0, y_0) = \left.\dfrac{dy}{dx}\right|_{(x_0,y_0)}$ and $(x_0,y_0) = (1,1)$ in this case. So $y - 1 = -\frac{9}{26}(x-1) \Rightarrow y = \frac{-9}{26}x + \frac{35}{26}$

So the approximation for the y-coordinate at $x = 1.05$ is $y = \frac{-9}{26}(1.05) + \frac{35}{26}$

$\Rightarrow y \sim 0.98$

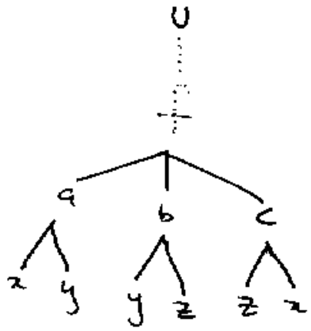
2 Let $F(x,y,z) = ye^z + xz - x^2 - y^2$

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{z - 2x}{ye^z + x}$ This is valid unless $ye^z + x = 0$. We can think of this region as the surface $z = \ln\left(-\frac{x}{y}\right)$ where $xy < 0$ together with the z-axis.

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{e^z - 2y}{ye^z + x}$ This one is valid in the same region as $\partial z/\partial x$.

$\boxed{3}$ Let $a = x-y$; $b = y-z$; $c = z-x$

$\boxed{\text{Solution 1}}$



A neat way to see the chain rule:
To compute $\frac{\partial u}{\partial x}$, use this tree and "add up" the branches that take you from u to x:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial x} = \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial c}(-1)$$

Similarly, we compute $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$:

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y} = \frac{\partial u}{\partial a}(-1) + \frac{\partial u}{\partial b}(1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial z} = \frac{\partial u}{\partial b}(-1) + \frac{\partial u}{\partial c}(1)$$

Then
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial a} - \frac{\partial u}{\partial c}\right) + \left(-\frac{\partial u}{\partial a} + \frac{\partial u}{\partial b}\right) + \left(-\frac{\partial u}{\partial b} + \frac{\partial u}{\partial c}\right) = 0.$$

$\boxed{\text{Solution 2}}$ We can also use Jacobean matrices:

$$\begin{bmatrix} u_x & u_y & u_z \end{bmatrix} = \begin{bmatrix} f_a & f_b & f_c \end{bmatrix}\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix} = \begin{bmatrix} f_a & f_b & f_c \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} f_a - f_c & -f_a + f_b & -f_b + f_c \end{bmatrix}$$

Thus $u_x + u_y + u_z = f_a - f_c - f_a + f_b - f_b + f_c = 0.$

$\boxed{4}$

$$J_F = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & -2z \\ y+z & x+z & x+y \end{bmatrix}$$

Now, $x = R\cos\theta\sin\phi$
$y = R\sin\theta\sin\phi$
$z = R\cos\phi$

Thus $J_F = \begin{bmatrix} R/2 & \sqrt{3}R/2 & -\sqrt{3}R \\ R\cdot\frac{3\sqrt{3}}{4} & \frac{R}{4}(1+2\sqrt{3}) & \frac{R}{4}(1+\sqrt{3}) \end{bmatrix}$ at $(\phi,\theta) = \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

$$J_G = \begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} R\cos\theta\cos\phi & -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi & R\cos\theta\sin\phi \\ -R\sin\phi & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4}R & -\frac{\sqrt{3}}{4}R \\ \frac{3}{4}R & \frac{1}{4}R \\ -\frac{R}{2} & 0 \end{bmatrix} \quad \text{at } (\phi,\theta)=\left(\tfrac{\pi}{6},\tfrac{\pi}{3}\right)$$

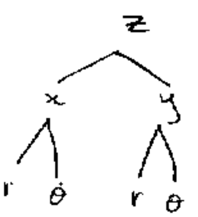
So

$$J_{F\circ G} = J_F \cdot J_G = \begin{bmatrix} R/2 & \sqrt{3}R/2 & -\sqrt{3}R \\ R\cdot\frac{3\sqrt{3}}{4} & \frac{R}{4}(1+2\sqrt{3}) & \frac{R}{4}(1+\sqrt{3}) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{4}R & -\frac{\sqrt{3}}{4}R \\ \frac{3}{4}R & \frac{1}{4}R \\ -\frac{R}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{8}R^2 + \frac{3\sqrt{3}}{8}R^2 + \frac{\sqrt{3}}{2}R^2 & -R^2\frac{\sqrt{3}}{8} + R^2\frac{\sqrt{3}}{8} \\ \frac{4}{16}R^2 + \frac{3}{16}R^2(1+2\sqrt{3}) - \frac{R^2}{8}(1+\sqrt{3}) & -R^2\frac{9}{16} + \frac{R^2}{16}(1+2\sqrt{3}) \end{bmatrix} = \begin{bmatrix} \sqrt{3}R^2 & 0 \\ \frac{5+2\sqrt{3}}{8}R^2 & -\frac{R^2}{2} + \frac{\sqrt{3}}{8}R^2 \end{bmatrix}$$

5 (a)

solution 1 We can use the "tree method" like in problem 3



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\sin\theta.$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta$$

solution 2 We can use Jacobean matrices

$$[z_r \quad z_\theta] = [z_x \quad z_y]\begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = [z_x x_r + z_y y_r \quad z_x x_\theta + z_y y_\theta]$$

$$= \left[\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \quad -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta\right]$$

from which we get

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \quad \text{and}$$

$$\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta.$$

(b) Take the right-hand side

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\right)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + 2\left(\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right)\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta$$

$$+ \left(\frac{\partial z}{\partial x}\right)^2 \sin^2\theta - 2\left(\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right)\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2\theta$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 (\cos^2\theta + \sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2 (\cos^2\theta + \sin^2\theta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = LHS \text{ as desired.}$$

6

(a) Let $f(x,y,z) = xy - x^2 + z^2 - 1$ and $g(x,y,z) = 2xz + y^3 - 3y$

Tangent plane to S_1 at $(1,1,1)$:

$(X - P_0)\cdot\vec{n} = 0$ where $X = (x,y,z)$, $P_0 = (1,1,1)$

$\vec{n} = \nabla f\big|_{(1,1,1)}$

$\downarrow$

$\vec{n} = (y-2x, x, 2z)\big|_{(1,1,1)} = (-1,1,2)$

$\Rightarrow ((x,y,z) - (1,1,1))\cdot(-1,1,2) = 0$

$\Rightarrow \boxed{-x + y + 2z = 2}$

Tangent plane to S_2 at $(1,1,1)$

$(X - P_0)\cdot\vec{n} = 0$ this time $\vec{n} = \nabla g\big|_{(1,1,1)} = (2z, 3y^2-3, 2x)\big|_{(1,1,1)} = (2,0,2)$

$\Rightarrow 2x + 2z = 4$

$\Rightarrow \boxed{x + z = 2}$

(b) The tangent line to C at $(1,1,1)$ is the intersection of the two planes calculated in part (a).

$$\left.\begin{aligned} -x + y + 2z &= 2 \\ x \qquad + z &= 2 \end{aligned}\right\}$$ direction vector $\vec{v}$ for the line is given by

$(-1,1,2)\times(2,0,2) = (2,6,-2) = 2(1,3,-1)$

Parametric equation is $\boxed{\ell(t) = P_0 + t\vec{v} = \begin{pmatrix}1\\1\\1\end{pmatrix} + t\begin{pmatrix}1\\3\\-1\end{pmatrix}}$

(c) Recall we defined $f(x,y,z) = xy - x^2 + z^2 - 1$

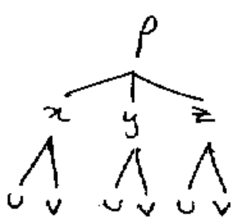
using implicit differentiation

$$\frac{\partial x}{\partial y} = \frac{-f_y}{f_x} = -\frac{x}{y-2x} \qquad \frac{\partial x}{\partial z} = -\frac{f_z}{f_x} = -\frac{2z}{y-2x}$$

$$= 1 \text{ at } (1,1,1) \qquad\qquad = 2 \text{ at } (1,1,1)$$

$\boxed{7}$

(a) I'll use the "tree method" for this one



$$\frac{\partial p}{\partial u} = \frac{\partial p}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial p}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= (3y)\left(\frac{1}{u}\right) + (3x+z)(\sin v) + (y)(v)$$

$$= \frac{3+3u\sin v}{u} + (3\ln u + 3\cos v + uv)\sin v + (1+u\sin v)v$$

$$= \boxed{\frac{3}{u} + 3\sin v + 3\ln u\sin v + 3\sin v\cos v + uv\sin v + v + uv\sin v}$$

$$\frac{\partial p}{\partial v} = \frac{\partial p}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial p}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial v} = (3y)(-\sin v) + (3x+z)u\cos v + (y)(u)$$

$$= (3+3u\sin v)(-\sin v) + (3\ln u + 3\cos v + uv)(u\cos v) + (1+u\sin v)(u)$$

$$= \boxed{-3\sin v - 3u\sin^2 v + 3u\ln u\cos v + 3u\cos^2 v + u^2 v\cos v + u + u^2\sin v}$$

at $(u,v) = (1,\pi)$ $\quad \cancel{\frac{\partial p}{\partial u}} \Rightarrow \frac{\partial p}{\partial u} = 3+\pi \qquad \frac{\partial p}{\partial v} = 4-\pi$

(b) using the chain rule in its basic form:

$$\frac{dp}{dt} = \frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial t} = \frac{\partial p}{\partial u}(\pi\cos(\pi t)) + \frac{\partial p}{\partial v}(2\pi t)$$

notice that at $t=1$, $(u,v) = (1,\pi)$, so from part (a)

$$\left.\frac{dp}{dt}\right|_{t=1} = (3+\pi)(\pi(-1)) + (4-\pi)(2\pi) = -3\pi - \pi^2 + 8\pi - 2\pi^2$$

$$= \boxed{5\pi - 3\pi^2}$$

8 (a) Say the dog is at the point (x, y)

It's distance (squared) to $(-5,-3)$ is $(x--5)^2+(y--3)^2$

similarly, the distances (squared) to the other two bones are

$(x-8)^2+(y-0)^2$ and $(x-0)^2+(y-7)^2$ respectively.

We wish to minimize their sum

$$(x+5)^2+(x-8)^2+x^2+(y+3)^2+(y-7)^2+y^2 = 3x^2-6x+3y^2-8y+147$$

Let $f(x,y) = 3x^2-6x+3y^2-8y+147$.

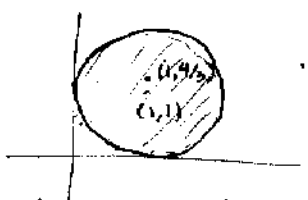
Find stationary points: $\begin{array}{l} f_x=0 \\ f_y=0 \end{array} \Rightarrow \left.\begin{array}{l} 6x-6=0 \\ 6y-8=0 \end{array}\right\} \Rightarrow x=1, y=4/3$

from a geometric argument, we see this point is a minimum.

If not convinced, we can calculate the determinant of the Hessian matrix $D = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0$ and $f_{xx} = 6 > 0 \Rightarrow$ min.

So the dog should sit at $(1, 4/3)$.

(b)

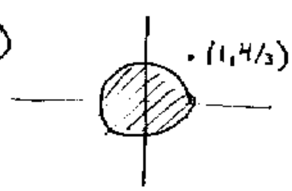


The dog is now contrained to the region $(x-1)^2+(y-1)^2 \le 1$

But our global min lies within the constrain region!

so the dog should still sit at $(1, 4/3)$.

(c)



The dog is now constrained to the region $x^2+y^2 \le 1$. Our global min is outside the region. Since our function has no other unconstrained extrema, we need only look for minima of f constrained to the boundary (i.e. the unit circle)

$f = 3x^2-6x+3y^2-8y+147$

$g = x^2+y^2-1$

Lagrange condition

$\nabla f = \lambda \nabla g$

$$\left.\begin{array}{l} 6x-6=\lambda 2x \\ 6y-8=\lambda 2y \end{array}\right\} \Rightarrow \left.\begin{array}{l} x(3-\lambda)=3 \\ y(3-\lambda)=4 \end{array}\right\}$$ Notice $x, y \neq 0$ so $\frac{3}{x}=\frac{4}{y} \Rightarrow y=\frac{4}{3}x$

$$x^2+y^2=1 \Rightarrow x^2+\frac{16}{9}x^2=1 \Rightarrow x^2=\frac{9}{25} \Rightarrow x=\pm\frac{3}{5} \quad y=\pm\frac{4}{5}$$

$$f\left(\frac{3}{5},\frac{4}{5}\right) = -\frac{181}{25}+147 \qquad f\left(-\frac{3}{5},-\frac{4}{5}\right) = \frac{319}{25}+147$$

$\Rightarrow$ The dog should sit at $\left(\frac{3}{5},\frac{4}{5}\right)$ since this minimizes f.

$\boxed{9}$ (a) Say our line of best fit is $ax+b$.

Let (x_i, y_i) be any one of the points given. The square of the vertical distance between this point and the line is

$$[y_i - (ax_i+b)]^2$$

We wish to minimize the sum of these (squared) distances:

Let $f(a,b) = \sum_{i=1}^{5} [y_i - (ax_i+b)]^2$ (Notice this is a function of a & b, not x and y!)

We want $\frac{\partial f}{\partial a}=0$ and $\frac{\partial f}{\partial b}=0$

$$\frac{\partial f}{\partial a} = \sum_{i=1}^{5} 2(y_i-ax_i-b)(-x_i) \quad \text{and} \quad \frac{\partial f}{\partial b} = \sum_{i=1}^{5} 2(y_i-ax_i-b)(-1)$$

$$\frac{\partial f}{\partial a}=0 \Rightarrow a\sum x_i^2 + b\sum x_i = \sum x_i y_i$$

$$\frac{\partial f}{\partial b}=0 \Rightarrow a\sum x_i + bm = \sum y_i$$

now solve these simultaneously

(m is the number of points we have)

$$\Rightarrow a = \frac{m\sum x_i y_i - (\sum x_i)(\sum y_i)}{m\sum x_i^2 - (\sum x_i)^2} \qquad b = \frac{(\sum x_i^2)(\sum y_i) - (\sum x_i y_i)(\sum x_i)}{m\sum x_i^2 - (\sum x_i)^2}$$

These expressions are ugly, but we make a little table to simplify our lives;

	x_i	y_i	x_i^2	$x_i y_i$
	-3	3	9	-9
	-2	3	4	-6
	0	2	0	0
	2	1	4	2
	4	-1	16	-4
Σ	1	8	33	-17

$$a = \frac{5(-17) - (1)(8)}{5(33) - (1)^2} = -\frac{93}{164}$$

$$b = \frac{(33)(8) - (-17)(1)}{5(33) - (1)^2} = \frac{281}{164}$$

So our line of best fit is $\boxed{\frac{-93}{164}x + \frac{281}{164}}$

(b) This time we won't do something as general as part (a) because the formulae get ugly very quickly.

We wish to minimize $f(a,b,c) = \sum_{i=1}^{5} [y_i - (ax_i^2 + bx_i + c)]^2$ where the (x_i, y_i) are the five given points.

You can verify that ~~f(a,b,c) =~~

$$f(a,b,c) = 24 - 54a + 369a^2 + 34b + 74ab + 33b^2 - 16c + 66ac + 2bc + 5c^2$$

We want $\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} = 0$

$$\left.\begin{aligned} f_a &= -54 + 738a + 74b + 66c = 0 \\ f_b &= 34 + 74a + 66b + 2c = 0 \\ f_c &= -16 + 66a + 2b + 10c = 0 \end{aligned}\right\}$$

Solving this system we find:

$a = -\frac{27}{388}$ $b = -\frac{195}{388}$ $c = \frac{419}{194}$

So the best fitting quadratic for these points is

$$\boxed{\frac{-27}{388}x^2 - \frac{195}{388}x + \frac{419}{194}}$$

NB This point is our minimum. We see this from a geometric argument. f is unbounded above i.e. we can make it as big as we want $\Rightarrow$ no maxima exists.

10. This problem is pretty neat. Let's think about it carefully before jumping into calculations. Let $\omega_1 = F(x,y,u,v) = c_1$ and $\omega_2 = G(x,y,u,v) = c_2$ where $U = f(x,y)$ and $V = g(x,y)$.
We can differentiate both sides of $\omega_1 = c$ w.r.t. x remembering that U and V depend on x.

$$\frac{\partial \omega_1}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} = 0.$$

We can write this more compactly as $F_x + F_u U_x + F_v V_x = 0$.
Differentiating $\omega_1 = c$ w.r.t y we get a similar equation: $F_y + F_u U_y + F_v V_y = 0$
We repeat this process with $\omega_2 = c$ and obtain another two equations. In the end, we'll have the following four equations:

$$\left.\begin{aligned} F_u U_x + F_v V_x &= -F_x \\ G_u U_x + G_v V_x &= -G_x \end{aligned}\right\} \qquad \left.\begin{aligned} F_u U_y + F_v V_y &= -F_y \\ G_u U_y + G_v V_y &= -G_y \end{aligned}\right\};$$

which we can re-write in the following convenient matrix form:

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} U_x \\ V_x \end{pmatrix} = -\begin{pmatrix} F_x \\ G_x \end{pmatrix} \qquad \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} U_y \\ V_y \end{pmatrix} = -\begin{pmatrix} F_y \\ G_y \end{pmatrix}$$

If we multiply both sets of equations by the inverse matrix of $\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}$ on the left, we'll have obtained expressions for U_x, V_x, U_y, V_y !! Recall the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

So

$$\begin{pmatrix} U_x \\ V_x \end{pmatrix} = \frac{-1}{F_u G_v - G_u F_v}\begin{pmatrix} G_v & -F_v \\ -G_u & F_u \end{pmatrix}\begin{pmatrix} F_x \\ G_x \end{pmatrix}$$

and
$$\begin{pmatrix} U_y \\ V_y \end{pmatrix} = \frac{-1}{F_u G_v - F_v G_u}\begin{pmatrix} G_v & -F_v \\ -G_u & F_u \end{pmatrix}\begin{pmatrix} F_y \\ G_y \end{pmatrix}$$

Thus

$$U_x = - \frac{G_v F_x - F_v G_x}{F_u G_v - G_u F_v} \quad \text{and} \quad V_y = - \frac{G_u F_y - F_u G_y}{F_u G_v - F_v G_u}.$$

Nice notation: notice the expressions on numerators and denominators look like determinants.

So let's write

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = F_u G_v - F_v G_u$$

Using this notation, our expressions become

$$U_x = - \frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} \qquad V_y = - \frac{\frac{\partial(F,G)}{\partial(y,u)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

and now it really looks like implicit differentiation in higher dimensions!

Finally, we remember that

$$F(x,y,u,v) = x^2 - y^2 - u^3 + v^2 + 4 = 0$$
$$G(x,y,u,v) = 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$

$$U_x = - \frac{(12v^3)(2x) - (2v)(2y)}{(-3u^2)(12v^3) - (-4u)(2v)} = \boxed{\frac{6xv^2 - y}{-9u^2v^2 + 2u}}$$

$$V_y = - \frac{(-4u)(-2y) - (-3u^2)(2x+2y)}{(-3u^2)(12v^3) - (-4u)(2v)} = \boxed{\frac{4y + 3u(x+y)}{-18uv^3 + 4v}}$$