PROBLEM SET #6 SOLUTIONS

1)

10. (a) $0 \le x \le 2, 0 \le y \le \sqrt{2x - x^2}, x^2 + y^2 + z^2 = 4$

    (b) $\int_0^{\pi/2} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta = 8\pi/3 - 32/9$

11. Using spherical coordinates, the volume of water in the bowl is $\int_0^{2\pi} \int_{\pi/2+\arcsin(3/5)}^{\pi} \int_{-3\sec\phi}^{5} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

2)

12. The height of the bowl is 3 inches:

$$99\pi = \int_0^{2\pi} \int_{\pi/2}^{\pi/2+\arcsin(h/6)} \int_0^6 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{2\pi} \int_{\pi/2+\arcsin(h/6)} \int_0^{-h\sec\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= 24\pi h + 12\pi h - \pi h^3/3 \implies h = 3.$$

NOTE: The other solutions of the cubic equation (i.e., $-3/2 \pm 9\sqrt{5}/2$ can be rejected on physical grounds).

3)

6. (a) The integral reduces to $\int_0^{\pi} (1 + \sin\theta)\, d\theta = \pi + 2$.

   (b) The surface lies above the unit circle in the xy-plane; the surface is defined by the function $g(r, \theta) = 1 - r^2$. Thus, in polar form, the integral is $\int_0^{2\pi} \int_0^1 (1 - r^2) r\, dr\, d\theta = \frac{\pi}{2}$.

   (c) The surface lies above the unit circle in the xy-plane; the surface is defined by the function $g(r, \theta) = 1 - r$. Thus, in polar form, the integral is $\int_0^{2\pi} \int_0^1 (1 - r) r\, dr\, d\theta = \frac{\pi}{3}$.

4)

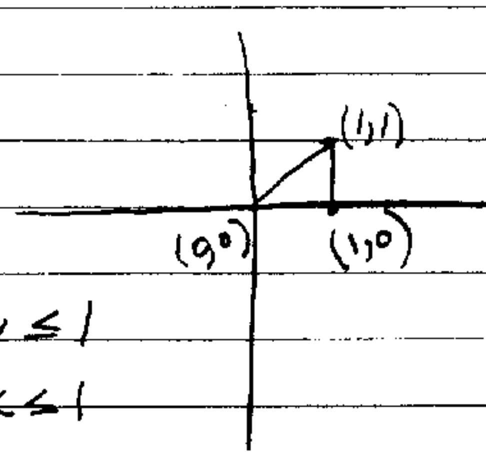
12. $\int_0^1 \int_{x/2}^{1/2} e^{y^2}\, dy\, dx = \int_0^{1/2} \int_0^{2y} e^{y^2}\, dx\, dy = \int_0^{1/2} 2y e^{y^2}\, dy = e^{1/4} - 1$

13. The calculation is the same as in the theorem's proof, but the order of integration is reversed.

5

WANT TO CALCULATE

$\iint 2e^{x^2}dA$ OVER R =



$0 \leq y \leq 1$

$y \leq x \leq 1$

WE'LL USE CARTESIAN COORDINATES SO

$$dA = dx\,dy \quad \text{or} \quad dy\,dx$$

INSPECTING THE INTEGRAND, WE SEE THAT WE NEED A FACTOR OF x TO COMPLETE THE INTEGRATION OVER x. THIS SUGGESTS THAT WE DO THE INTEGRATION OVER y FIRST, AND HOPE THAT IT PRODUCES THE FACTOR WE NEED.

THUS, WE NEED TO CHANGE OUR LIMIT DEFINITION

BY EXAMINING OUR PICTURE WE CAN TRANSFORM:

$$0 \le y \le 1, \; y \le x \le 1$$

INTO

$$0 \le x \le 1, \; 0 \le y \le x$$

GIVING US:

$$\int_0^1 \int_0^x 2e^{x^2} \, dy \, dx$$

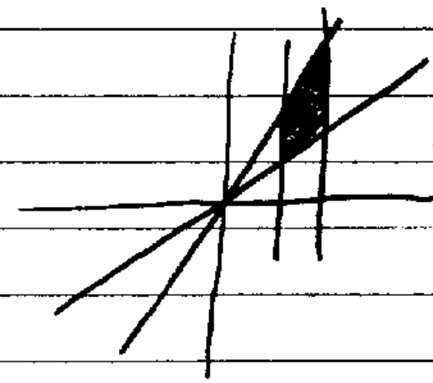
THE REST IS NOW STRAIGHT FORWARD:

$$\int_0^1 2xe^{x^2} \, dx$$

$$e^{x^2} \Big|_0^1 = \boxed{e - 1}$$

PROBLEM

$$\iint_R x/y \, dA \qquad R =$$



DEFINED BY

$x \le y \le 2x$

$1 \le x \le 2$

$dA = dx\,dy$ or $dy\,dx$

JUST SET IT UP AND GO:

$$\int_1^2 \int_x^{2x} x/y \; dy\,dx$$

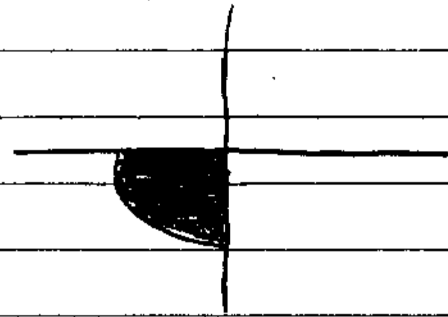
y INTEGRAL FIRST TO BRING x DOWN

$$\int_1^2 x\left[\ln(2x) - \ln(x)\right] dx = \int_1^2 x \ln 2 \, dx$$

$$\int_1^2 x \ln 2 \, dx = \frac{x^2}{2} \ln 2 \Big|_1^2 = \boxed{\tfrac{3}{2} \ln 2}$$

7. INTEGRATE:

$$f(x,y) = \frac{2}{1+\sqrt{x^2+y^2}} \text{ ON } R.$$



WHERE R IS DEFINED BY

$$\left.\begin{array}{l} -1 \le x \le 0 \\ -(1-x^2) \le y \le 0 \end{array}\right\}$$

POLAR COORDINATES ARE AN OBVIOUS CHOICE

THEN WE DEFINE OUR REGION:

$$\left.\begin{array}{l} 0 \le r \le 1 \\ \pi \le \theta \le 3\pi/2 \end{array}\right\}$$

AND THE INTEGRAL TAKES THE FORM:

$$\int_0^1 \int_\pi^{3\pi/2} \frac{2r}{1+r} \, d\theta \, dr$$

THIS IS <u>MUCH</u> MORE MANAGEABLE

5

Θ IS TRIVIAL

$$\int_0^1 \int_{\pi}^{3\pi/2} \frac{2r}{1+r}\, d\theta\, dr = \pi/2 \underbrace{\int_0^1 \frac{2r}{1+r}\, dr}$$

LOOK THIS UP IN A TABLE,
OR USE MATHEMATICA
[or just do the division]

$$\pi/2 \int_0^1 \frac{2r}{1+r}\, dr = \pi/2 \left[2\left(r - \ln(r+1)\right)\right]\Big|_0^1$$

$$\boxed{= \pi\left(1 - \ln 2\right)}$$

(6)

~~8~~

$\cos 3\theta$ HAS A PERIOD OF $2\pi/3$,

SO AN INTEGRAL TO HALF THAT WILL

GIVE US ONE LEAF:

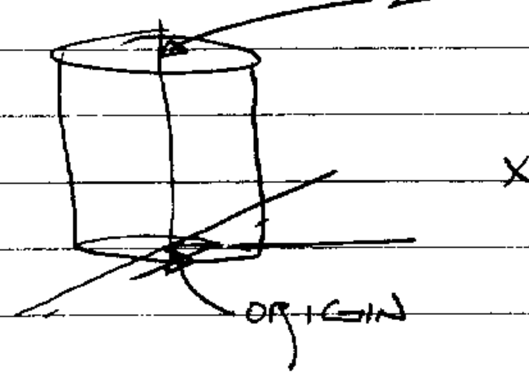
$$\int_0^{\pi/3} \int_0^{12\cos 3\theta} r\,dr\,d\theta$$

r FIRST TO PULL DOWN θ

$$72\int_0^{\pi/3} \cos^2 3\theta \, d\theta = 24 \int_0^{\pi} \cos^2\theta \, d\theta$$

$$\boxed{= 12\pi}$$

9 HERE'S OUR CYLINDER:



$0 \le z \le 10$

$x^2+y^2 \le 1$

PROBLEMS ON CYLINDERS ARE USUALLY A GOOD TIME TO USE CYLINDRICAL COORDINATES.

DUH.

SO, OUR NEW LIMITS ARE:

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$0 \le z \le 10$

(A) WE'LL OF COURSE USE

$$\text{MASS} = \int \text{DENSITY} \cdot d\,\text{VOLUME}$$

OUR DENSITY FUNCTION, IN CYLINDRICAL COORDINATES:

$$\sigma(r, \theta, z) = (100 - z^2)(1 - r^2)$$

$$dV = r\,dr\,d\theta\,dz \ \underline{\underline{\text{or}}}\ r\,d\theta\,dz\,dr$$

(or 4 other permutations)

LET'S ROCK'N'ROLL!

$$M = \int_0^1 \int_0^{10} \int_0^{2\pi} (100 - z^2)(1 - r^2)\, r\, d\theta\, dz\, dr$$

GET θ OUT OF THE WAY:

$$M = 2\pi \int_0^1 \int_0^{10} (100 - z^2)(1 - r^2)\, r\, dz\, dr$$

NOW Z:

$$M = 2\pi \int_0^1 (1-r^2)\,r\,dr \left[100z - z^3/3\right]\Big|_0^{10}$$

$$= \frac{4000}{3}\pi \int_0^1 (1-r^2)\,r\,dr$$

LASTLY r:

$$= 4000/3\,\pi \left[r^2/2 - r^4/4\right]\Big|_0^1$$

$$= \boxed{1000/3\,\pi \text{ UNITS}^3}$$

(B) BECAUSE OF SYMMETRY, WE EXPECT THE C.O.M. TO BE ON THE CYLINDRICAL AXIS OF THE CYLINDER. THEREFORE, WE'LL ONLY COMPUTE $\bar{Z}$.

$$\bar{Z} = \frac{1}{M_{TOT}} \int_0^1 \int_0^{10} \int_0^{2\pi} z\left(100 - z^2\right)\left(1 - r^2\right) r \, d\theta \, dz \, dr$$

PART (A)

θ IS EASY, AS USUAL

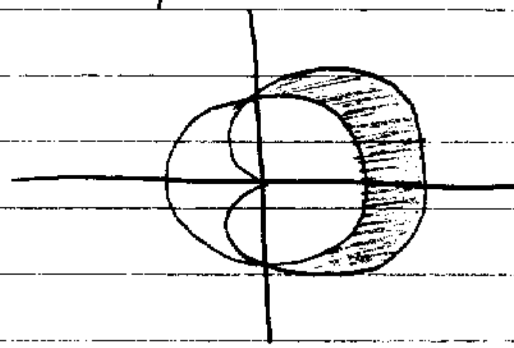
$$= \frac{3}{500} \int_0^1 \int_0^{10} z\left(100 - z^2\right)\left(1 - r^2\right) r \, dz \, dr$$

THE MATH IS SIMILAR TO (A). CUTTING TO THE CHASE:

$$\bar{Z} = \frac{15}{4}$$

$\boxed{\text{SO, } (0,0,15/4) \text{ IS THE C.O.M IN CARTESIAN COORDS}}$

THE PICTURE LOOKS VAGUELY THUS:



WE'RE INTERESTED IN THE SHADED REGION,

SO

$$1 \le r \le 1 + \cos\theta$$

$$0 \le z \le r\cos\theta$$

$$-\pi/2 \le \theta \le \pi/2$$

NOTICE THAT THIS IS ALSO WHERE THE HEIGHT IS POSITIVE, WHICH IS GOOD.

SO, WE SIMPLY WANT TO INTEGRATE

$$dV = r\,dr\,d\theta\,dz \quad \text{or} \quad r\,dz\,dr\,d\theta$$

OVER THE REGION DEFINED.

THUS, OUR INTEGRAL

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} \int_{0}^{r\cos\theta} r \, dz \, dr \, d\theta$$

Z INTEGRAL BRINGS DOWN r & θ, SO

DO IT FIRST

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} r^2 \cos\theta \, dr \, d\theta$$

R INTEGRAL NEXT:

$$\frac{1}{3} \int_{-\pi/2}^{\pi/2} \left[(1+\cos\theta)^3 - 1 \right] d\theta$$

THE REST IS MESSY, BUT STRAIGHTFORWARD.

MATHEMATICA GIVES: $\boxed{V = \frac{4}{3} + \frac{5}{8}\pi}$