Math 21a - Spring 2001 - Problem Set #7

Dates covered: Apr 6 - Apr 18
Text sections: 3.4, 3.5, (5.1-5.2), 5.3
Key topics: Multiple integrals in cylindrical and spherical coordinates; center of mass and other applications of integrals; general change of variables in multiple integrals and Jacobian determinants; Fundamental Theorem of Line Integrals; Green's theorem.

(1) A solid ice cream cone is bounded on top by the sphere of radius *a* below by the cone with angle $\phi = \pi/6$. Find the gravitational attraction of the ice cream cone on a unit mass at its vertex, i.e. at the origin, if (a) the density is 1; and (b) the density is given by the distance from the vertex.

[**Note**: The gravitational attraction between two point masses m_1 and m_2 is given by $\mathbf{F} = \frac{Gm_1m_2}{\rho^2}\mathbf{u}$ where ρ is the distance between the points and **u** is a unit vector pointing from one mass toward the other. In this case, we can take $m_1 = 1$ and use Riemann Sums to construct a triple integral in spherical coordinates for the *z*-component of the gravitational force. The other components will cancel by symmetry.]

(2) Ostebee-Zorn problem 3.5/12.

(3) Evaluate the integral $\iint_R \cos\left(\frac{x-y}{x+y}\right)dxdy$ over the triangular region *R* with vertices at (0, 0), (1, 0), and (0, 1) by making use of the change of coordinates given by $x = \frac{u+v}{2}$, $y = \frac{-u+v}{2}$.

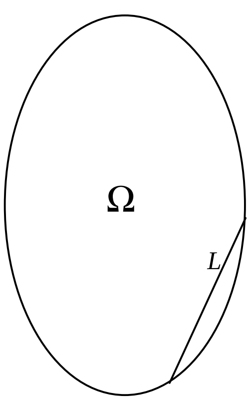
(4) The surface of a mountain is given by $z = 16 - x^2 - 4y^2$, where $x^2 + 4y^2 \le 16$. Find the volume inside the mountain, i.e. under its surface and with $z \ge 0$. [Hint: Try scaling one of the coordinates.]

(5) Find the center of mass of the plane region bounded by the three curves $y = 2x$, $y = x$, and $xy = 3$. Assume the density is constant.

(6) Ostebee-Zorn problems 5.3/8

(7) Ostebee-Zorn problems 5.3/10

Super Challenge Problem:



Let Ω be a convex region in $\mathbf{R}^2$ and let L be a line segment of length I that connects points on the boundary of Ω. As we move one end of L around the boundary, the other end will also move about this boundary, and the midpoint of L will trace out a curve within Ω that bounds a (smaller) region Γ. Find an expression that relates the area of Γ to the area of Ω in terms of the length I of the line segment. [Partial Hint: Use the Green's Theorem result that allows you to calculate the area of a region in terms of a line integral around the boundary of the region.]

You may use any resources you wish to solve this problem, but you should give credit where credit is due.