Solution Set #7: Math 21a

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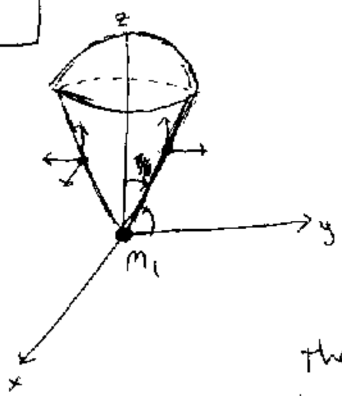

Clearly all x and y components of force will cancel out by symmetry, but we can show this as well:

$F = \frac{G m_1 m_2}{\rho^2}\vec{u}$ where m_1 is at the origin and m_2 is an infinitesimal mass element of the cone

where σ is the density function: hence $m_2 = \sigma(x,y,z)\,dx\,dy\,dz$

The total force is equal to the sum of all component forces by the superposition principle

hence $\vec{F}_{tot} = \sum$ all forces $= \lim_{dxdydz\to 0} \sum_{cone} \frac{Gm_1}{\rho^2}\sigma(x,y,z)\,dx\,dy\,dz \cdot \vec{u}(x,y,z) = \iiint_{cone} \frac{Gm_1}{\rho^2}\sigma(x,y,z)\,dx\,dy\,dz\,\vec{u}$

However, we already know that this integral is much easier in spherical coordinates, so we make the change of variables: $(x,y,z) \to (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$ and the determinant of the Jacobian is $\rho^2\sin\phi$ hence our original integral becomes

$$\iiint_{cone} \frac{Gm_1}{\rho^2}\sigma(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,|\rho^2\sin\phi|\,d\rho\,d\phi\,d\theta\cdot\vec{u}$$

actually should be here

where u is the unit vector pointing in the direction of $\vec{F}$; and clearly $u = \frac{(x,y,z)}{(x^2+y^2+z^2)^{1/2}} = \frac{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)}{(\rho^2)^{1/2}}$;

The limits on the integral are: $0 \le \rho \le a$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{6}$

hence the integral is $\vec{F}_{total} = \int_0^{\pi/6}\int_0^{2\pi}\int_0^{a} Gm_1\sigma(\rho\sin\phi\cos\theta, \cdots)\sin\phi\,(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\,d\rho\,d\theta\,d\phi$

and clearly, since $\int_0^{2\pi}\sin\theta\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta = 0$ and σ is indep of θ in both cases the x and y components go to zero and $F_z = \int_0^{\pi/6}\int_0^{2\pi}\int_0^{a} Gm_1\sigma(\vec{X})\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi$; $\vec{X} = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

a) $\sigma(\vec{X}) = 1 \Rightarrow F_z = \int_0^{\pi/6}\int_0^{2\pi}\int_0^{a} Gm_1\frac{\sin 2\phi}{2}\,d\rho\,d\theta\,d\phi = \int_0^{\pi/6}\int_0^{2\pi}\frac{aGm_1}{2}\sin 2\phi\,d\theta\,d\phi = \int_0^{\pi/6} a\pi Gm_1\sin 2\phi$

$= a\pi Gm_1\left[-\frac{1}{2}\cos 2\phi\right]_0^{\pi/6} = a\pi Gm_1\left[-\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\right] = \boxed{\frac{Gm_1 a\pi}{4}}$

1 cont. b, here $\sigma(X) = \rho$

$$\therefore F_z = \int_0^{\pi/6}\int_0^{2\pi}\int_0^{a} 6m_1\rho\frac{\sin 2\phi}{2}\,d\rho\,d\theta\,d\phi = \int_0^{\pi/6}\int_0^{2\pi}\frac{a^2}{4}6m_1\sin 2\phi\,d\theta\,d\phi$$

$$= \frac{a^2\pi 6m_1}{2}\left[-\tfrac{1}{2}\cos 2\phi\right]_0^{\pi/6} = \boxed{\frac{6m_1 a^2\pi}{8}}$$

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2 $1 \le x \le 2$ hence x is nonzero on D thus $0 \le xy \le 2 \equiv 0 \le y \le 2/x$

and we know that $0 \le z \le 1$ hence the integral is

$$\int_0^1\int_1^2\int_0^{2/x}(x^2y + 3xyz)\,dy\,dx\,dz = \int_0^1\int_1^2\left[\frac{x^2y^2}{2} + \frac{3xy^2z}{2}\right]_0^{2/x} = \int\int \frac{x^2}{2}\cdot\frac{4}{x^2} + \frac{3x\cdot\frac{4}{x^2}\cdot z}{2}$$

$$= \int_0^1\int_1^2\left(2 + \frac{6z}{x}\right)dx\,dz = \int_0^1\left[2x + 6z\ln x\right]_1^2 = \int_0^1(4 - 2 + 6z\ln 2 - 0)\,dz = \int_0^1(2 + 6\ln 2\,z)\,dz =$$

$$\left[2z + 3\ln 2\, z^2\right]_0^1 = \boxed{2 + 3\ln 2}$$

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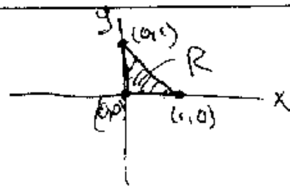
3 Okay, so our region is

$$\iint_R \cos\left(\frac{x-y}{x+y}\right)dx\,dy$$



the line through $(0,1)$ and $(1,0)$ is $y = mx + b$; $m = \frac{1-0}{0-1} = -1$ and
$y = -x + b$ plugging in $0,1 \Rightarrow 1 = b$
$y = -x + 1$

so in our region $0 \le y \le 1 - x$ $0 \le x \le 1$

now that we have our limits, it seems pretty obvious that a change of variables is in order, because neither our limits nor the expression $\cos\left(\frac{x-y}{x+y}\right)$ make this integral doable as it is. Let's use the COV suggested by the problem: $x = \frac{u+v}{2}$; $y = \frac{-u+v}{2}$

hence our Jacobian is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and the determinant $= \boxed{\frac{1}{2}}$

$x - y = u$ and $x + y = v$ hence our integral becomes $\iint \cos\left(\frac{u}{v}\right)\frac{1}{2}\,du\,dv$

the only slightly tricky thing is figuring out our limits: R is the triangle formed by $(0,0)$ $(0,1)$ and $(1,0)$ and under this COV, R becomes a new triangle in the u-v plane R' where the vertices are $x = 0 = \frac{u+v}{2}$, $y = 0 = \frac{v-u}{2}$; $x = 0 = \frac{u+v}{2}$, $y = 1 = \frac{v-u}{2}$; $x = 1 = \frac{u+v}{2}$, $y = 0 = \frac{v-u}{2}$ because our COV is linear

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R: $1 \le x \le 2$, $0 \le xy \le 2$, $0 \le z \le 1$ use cov $x = u$, $xy = v$, $3z = w$ $\Rightarrow$ $1 \le u \le 2$, $0 \le v \le 2$, $0 \le w \le 3$; $u \neq 0 \Rightarrow y = \frac{v}{u}$

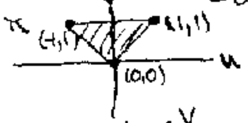
Jacobian: $\begin{bmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$ det $= \frac{1}{3u}$

$$\iiint_R (x^2y + 3xyz)\,dx\,dy\,dz = \int_1^2\int_0^3\int_0^2 (uv + vw)\frac{1}{3u}\,dv\,dw\,du$$

$$= \int_1^2\int_0^3 \left[uv^2/2 + \frac{v^2w}{2}\right]_0^2 \frac{1}{3u}\,dw\,du = \int_1^2\int_0^3 \frac{1}{3u}\left[2u + 2w\right]dw\,du = \int_1^2 \frac{1}{3u}\left[2uw + w^2\right]_0^3 du$$

$$= \int_1^2 \frac{1}{3u}\left[6u + 9\right]du = \int_1^2 \left[2 + \frac{3}{u}\right]du = \left[2u + 3\ln u\right]_1^2 = \boxed{2 + 3\ln 2}$$

3 cont) these equations yield $u+v=0,\ v-u=0 \Rightarrow v=0,\ u=0$; $u+v=0,\ v-u=2 \Rightarrow v=1,\ u=-1$; $u+v=2,\ v-u=0 \Rightarrow v=1,\ u=1$

Hence the new region is 

the limits are obviously
$-v \le u \le v$
$0 \le v \le 1$

hence our integral becomes $\frac{1}{2}\int_0^1\int_{-v}^{v}\cos\left(\frac{u}{v}\right)du\,dv$; $\int\cos\left(\frac{u}{v}\right)du = +v\sin\left(\frac{u}{v}\right)$

hence $\int_{-v}^{v}\cos\left(\frac{u}{v}\right)du = +v\sin\left(\frac{u}{v}\right)\Big]_{-v}^{v} = +v\sin(1) - v\sin(-1) = 2v\sin(1)$

$\frac{1}{2}\int_0^1 2v\sin(1)\,dv = v^2\sin(1)\Big]_0^1 \frac{1}{2} = \boxed{\frac{\sin(1)}{2}}$

So answer $= \frac{\sin(1)}{2}$

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4) okay, so $0 \le z \le 16 - x^2 - 4y^2$; $x^2 + 4y^2 \le 16$; ~~now let~~

now, let $z = z$, $x = x$ and $y = \frac{u}{2}$ then $0 \le z \le 16 - x^2 - u^2$ and $x^2 + u^2 \le 16$

thus the integral $\iiint dx\,dy\,dz \Rightarrow \iiint \frac{1}{2}dx\,du\,dz$ but if we now convert to cylindrical

the integral is easy; $z = z$, $x = r\cos\theta$, $u = r\sin\theta$ $\quad 0 \le z \le 16 - r^2 \quad r^2 \le 16$

by definition $0 \le \theta \le 2\pi$

hence $0 \le z \le 16 - r^2$; $0 \le r \le 4$ since r is positive

thus the integral becomes $\int_0^{2\pi}\int_0^4\int_0^{16-r^2}\frac{r}{2}\,dz\,dr\,d\theta$; $0 \le \theta \le 2\pi$ is obvious if you draw the picture;

thus integral $= \int_0^{2\pi}\int_0^4 \frac{r}{2}(16 - r^2)\,dr\,d\theta = \int_0^{2\pi}\left[4r^2 - \frac{r^3}{6}\right]_0^4 d\theta = \int_0^{2\pi}\frac{160}{3}d\theta = \boxed{\frac{320\pi}{3}}$

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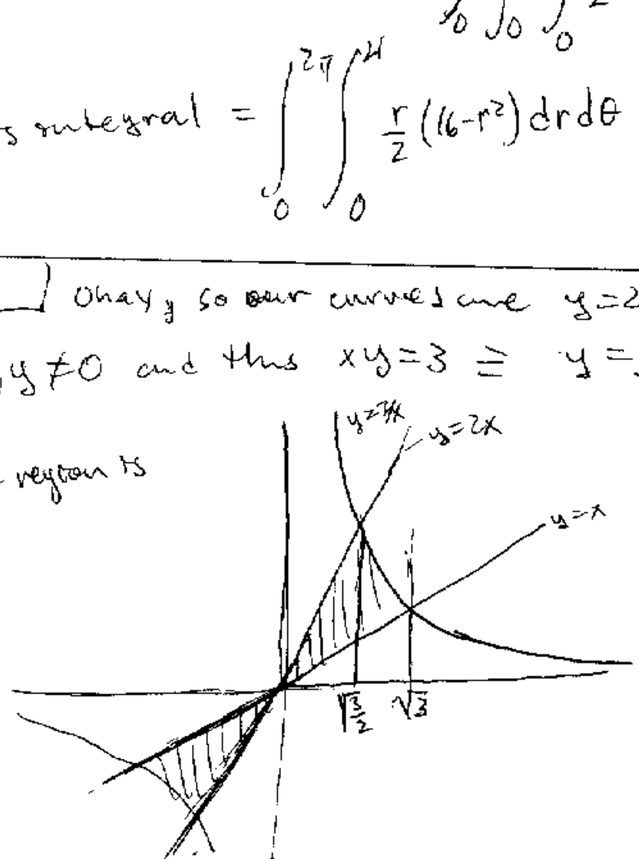
5) Okay, so our curves are $y = 2x$, $y = x$ and $xy = 3$; and since $xy = 3 \ne 0$

$x, y \ne 0$ and thus $xy = 3 \Rightarrow y = \frac{3}{x}$

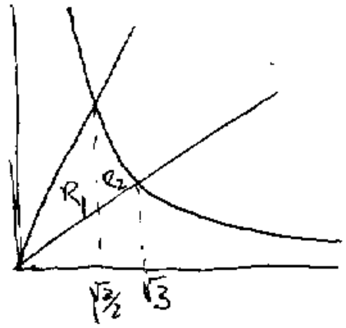
the region is 

the curve $y = \frac{3}{x}$ intersects $y = x$ at $x = \pm\sqrt{3}$

and the curve $y = \frac{3}{x}$ intersects $y = 2x$ at $x = \pm\sqrt{\frac{3}{2}}$, thus region is

And just by looking at the picture, this region is symmetric w/respect to the origin; just think about it, and you'll see that the center of mass is clearly $(0,0)$ HOWEVER

We were supposed to assume that the region was only in the first quadrant making the integral a little bit harder, since we actually have to evaluate it, which we can evaluate by splitting up the integral into R_1 and R_2



R_1	R_2
$x \le y \le 2x$	$x \le y \le 3/x$
$0 \le x \le \sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}} \le x \le \sqrt{3}$

$$\text{Mass} = \int_0^{\sqrt{\frac{3}{2}}}\int_x^{2x} dy\,dx + \int_{\sqrt{3/2}}^{\sqrt{3}}\int_x^{3/x} dy\,dx$$

$$\int_0^{\sqrt{\frac{3}{2}}} x\,dx = \frac{3}{4} \qquad \int_{\sqrt{\frac{3}{2}}}^{\sqrt{3}} \frac{3}{x} - x\,dx = 3\ln x - \frac{x^2}{2}\Big]_{\sqrt{3/2}}^{\sqrt{3}} = 3\ln(\sqrt{3}) - 3\ln\left(\sqrt{\tfrac{3}{2}}\right) - \frac{3}{2} + \frac{3}{4}$$

$$= 3\ln(\sqrt{2}) - \frac{3}{4} = \frac{3}{2}\ln(2) - \frac{3}{4}$$

$$\Rightarrow \boxed{\text{mass} = \frac{3}{2}\ln(2)}$$

$$x_{cm} = \frac{1}{M}\int_0^{\sqrt{\frac{3}{2}}}\int_x^{2x} x\,dy\,dx = \int_0^{\sqrt{3/2}} x^2\,dx = \frac{1}{2}\sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{4}\Big/M$$

$$+\frac{1}{M}\int_{\sqrt{3/2}}^{\sqrt{3}}\int_x^{3/x} x\,dy\,dx = 3x - \frac{x^3}{3}\Big]_{\sqrt{3/2}}^{\sqrt{3}} = 3\sqrt{3} - 3\sqrt{\frac{3}{2}} - \frac{3\sqrt{3}}{3} + \frac{3\sqrt{3}}{3\sqrt{2}} = 2\sqrt{3} - \frac{5\sqrt{6}}{4}$$

$$\Rightarrow x_{cm} = \left(2\sqrt{3} - \frac{\sqrt{6}}{\cancel{4}}\right)\Big/\frac{3}{2}\ln(2) = \frac{4\sqrt{3}}{3\ln(2)} - \frac{2\sqrt{6}}{3\ln 2}$$

$$y_{cm} = \frac{1}{M}\int_0^{\sqrt{\frac{3}{2}}}\int_x^{2x} y\,dy\,dx = \frac{1}{M}\int_0^{\sqrt{\frac{3}{2}}} y^2/2\Big]_x^{2x} = \int_0^{\sqrt{\frac{3}{2}}}\frac{3x^2}{2} = \frac{x^3}{2}\Big]_0^{\sqrt{\frac{3}{2}}} = \frac{1}{2}\cdot\frac{3}{2}\cdot\sqrt{\frac{3}{2}} = \frac{3}{4}\sqrt{\frac{3}{2}}\Big/M = \frac{3\sqrt{6}}{8M}$$

$$+\frac{1}{M}\int_{\sqrt{3/2}}^{\sqrt{3}}\int_x^{3/x} y\,dy\,dx = \int_{\sqrt{\frac{3}{2}}}^{\sqrt{3}} y^2/2\Big]_x^{3/x} = \int_{\sqrt{\frac{3}{2}}}^{\sqrt{3}}\frac{9}{2x^2} - \frac{x^2}{2}\,dx = -\frac{9}{2x} - \frac{x^3}{6}\Big]_{\sqrt{\frac{3}{2}}}^{\sqrt{3}} = \frac{-9}{2\sqrt{3}} - \frac{3\sqrt{3}}{6} + \frac{9}{2\sqrt{\frac{3}{2}}} + \frac{1}{6}\cdot\frac{3}{2}\cdot\sqrt{\frac{3}{2}}$$

$$= -\frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{6} + \frac{3\sqrt{6}}{2} + \frac{\sqrt{6}}{8}$$

$$y_{cm} = \left(2\sqrt{6} - 2\sqrt{3}\right)/M = \frac{4\sqrt{6}}{3\ln 2} - \frac{4\sqrt{3}}{3\ln 2}$$

$$\boxed{cm = \left(\frac{4\sqrt{3} - 2\sqrt{6}}{3\ln 2}, \frac{4\sqrt{6} - 4\sqrt{3}}{3\ln 2}\right)}$$

6) γ is the unit circle, so the simplest parameterization is $\gamma(t) = (\cos t, \sin t)$ $0 \le t \le 2\pi$ and $\gamma'(t) = (-\sin t, \cos t)$ hence $\big(P(\gamma(t)), Q(\gamma(t))\big) = (-\sin t, \cos t)$

a) $\displaystyle\int P\,dx + Q\,dy = \int_0^{2\pi} \sin^2 t + \cos^2 t = 2\pi$

b) (P,Q) is not a gradient field; if it were, the line integral around the closed loop γ, would equal zero

c) I think that you can copy the book's example SOX here, so I won't go into detail on this one: since $Q_x - P_y = 0$ everywhere on $\mathbb{R}^2$ where $(x,y) \ne (0,0)$ hence by the fact on page 272 $\displaystyle 0 = \iint_R (Q_x - P_y)\, dA = \oint_\gamma P\,dx + Q\,dy - \oint_S P\,dx + Q\,dy$

$\displaystyle \Rightarrow \oint_\delta P\,dx + Q\,dy = \oint_\gamma P\,dx + Q\,dy = 2\pi$; R is the region between the square and γ.

d) Green's theorem didn't work on γ because it contained the origin where $Q_x - P_y$ was undefined, however, ~~at the~~ on a circle that does not contain the origin it does and since $Q_x - P_y = 0$ the line integral over any circle not containing $(0,0) = 0$; the line integral around any closed loop not containing the origin $= 0$ because $Q_x - P_y = 0$, but the line integral around the origin over $\gamma = 2\pi$ because the line integral around the origin over $\gamma = 2\pi$ (P,Q) is still not a gradient

7) this question is pretty cool, but unfortunately it's not as simple as the last one and we actually have to do some work to get the answer:

First, it seems pretty obvious that we are supposed to use Green's theorem, and all of these points form an n-sided polygon in $\mathbb{R}^2$. Our goal is to take the line integral around this polygon and relate that to the area of the polygon: Well, we can parameterize each side of the polygon and then take the line integral around the entire polygon piece by piece, and add it all up; so let γ_k be the kth line segment between (x_k, y_k) and (x_{k+1}, y_{k+1}); then the parametrization of γ_k is just that of a line segment between (x_k, y_k) and (x_{k+1}, y_{k+1})

$\therefore \gamma_k(t) = (x_k, y_k) + t(x_{k+1} - x_k, y_{k+1} - y_k)$; $0 \le t \le 1$ hence, the line integral of f over γ_k is $\displaystyle\int_0^1 f(\gamma_k(t)) \cdot \gamma_k'(t)\, dt$ and $\gamma_k'(t) = (x_{k+1} - x_k, y_{k+1} - y_k)$ So if γ is the whole polygon

$$\oint_\gamma f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^1 f(\gamma_1(t)) \cdot \gamma_1'(t)\, dt + \int_0^1 f(\gamma_2(t)) \cdot \gamma_2'(t)\, dt + \cdots + \int_0^1 f(\gamma_n(t)) \cdot \gamma_n'(t)\, dt$$

$$= \sum_{k=1}^{n} \int_0^1 f(\gamma_k(t)) \cdot \gamma_k'(t)\, dt$$

7 cont.) In order to relate this to the area, we had better pick $f(x,y) = (P,Q)$

s.t. $Q_x - P_y = \text{const} \neq 0$ hence $\oint_\gamma P\,dx + Q\,dy = \text{const} \cdot$ Area of Polygon

Well, if $(P,Q) = (0,x)$ then $Q_x - P_y = 1$ and hence $\oint_\gamma P\,dx + Q\,dy =$ Area of polygon

But $(0,x)$ is only one possible (P,Q) ~~and~~ it's the lazy man's parameterization given that

zero, but I'm lazy, and a man so I'll use it:

$$\Rightarrow \int_0^1 f(\gamma_k(t)) \cdot \gamma'(t)\,dt = \int_0^1 (0, x_k + t(x_{k+1} - x_k)) \cdot (x_{k+1} - x_k, y_{k+1} - y_k)\,dt = \int_0^1 x_k y_{k+1} - x_k y_k + t x_{k+1} y_{k+1} - t x_{k+1} y_k - t x_k y_{k+1} + t x_k y_k\,dt = \left[x_k y_{k+1} - x_k y_k + \frac{x_{k+1} y_{k+1}}{2} - \frac{x_{k+1} y_k}{2} - \frac{x_k y_{k+1}}{2} + \frac{x_k y_k}{2}\right]$$

$$= \frac{1}{2}\left[x_k y_{k+1} + x_{k+1} y_{k+1} - x_{k+1} y_k + x_k y_k\right] = \frac{1}{2}\left[(x_{k+1} + x_k)(y_{k+1} - y_k)\right]$$

$$\text{Hence Area} = \sum_{k=1}^{n} \int_0^1 f(\gamma_k(t)) \cdot \gamma'(t)\,dt = \sum_{k=1}^{n} \frac{1}{2}\left[(x_{k+1} + x_k)(y_{k+1} - y_k)\right] = \frac{1}{2} \sum_{k=1}^{n} \left((x_{k+1} + x_k)(y_{k+1} - y_k)\right) \blacksquare$$

A Solution to the Super Challenge Problem

Let Ω be a convex region in $\mathbf{R}^2$ and let L be a line segment of length I that connects points on the boundary of Ω. As we move one end of L around the boundary, the other end will also move about this boundary, and the midpoint of L will trace out a curve within Ω that bounds a (smaller) region Γ. Find an expression that relates the area of Γ to the area of Ω in terms of the length I of the line segment.

Solution: Denote by $\mathbf{x}_1$ and $\mathbf{x}_2$ the two points where the segment contacts the boundary of Ω.

(1) Note that $\mathbf{x} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ is the midpoint of the segment. If we denote by $\mathbf{u}$ the vector of length I that connects $\mathbf{x}_1$ to $\mathbf{x}_2$, we also have $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{u}$. In components we have $\left\{\begin{matrix} x_2 = x_1 + u_x, \quad y_2 = y_1 + u_y \\ x = \frac{1}{2}(x_1 + x_2),\ y = \frac{1}{2}(y_1 + y_2) \end{matrix}\right\}$.

(2) If we initially parametrize $\mathbf{x}_1$ as it moves about the boundary of Ω, i.e. $\partial\Omega$, as $\mathbf{x}_1(t)$ where $t \in [0,1]$, then implicitly we can also use this same parameter to parametrize the 2nd point and the midpoint. Thus we have $\mathbf{x}_2(t) = \mathbf{x}_1(t) + \mathbf{u}(t)$. Note that $\mathbf{x}_2(t)$ will also trace out $\partial\Omega$ and $\mathbf{u}(t)$ will trace out the boundary of a disk of radius I as t goes from 0 to 1.

(3) By Green's Theorem, we know that $\int_{\partial\Omega} x_1 dy_1 = \int_{\partial\Omega} x_2 dy_2 = area(\Omega) = A$. We also know that $\int_{\partial\Gamma} x\,dy = area(\Gamma)$. If we use the relations in (1) and the parametrization in (2), we get the following:

$$area(\Gamma) = \int_{\partial\Gamma} x\,dy = \int_0^1 \tfrac{1}{2}(x_1(t) + x_2(t))\tfrac{1}{2}\left(\frac{dy_1}{dt} + \frac{dy_2}{dt}\right)dt$$

$$= \tfrac{1}{4}\left[\int_0^1 x_1(t)\frac{dy_1}{dt}dt + \int_0^1 x_1(t)\frac{dy_2}{dt}dt + \int_0^1 x_2(t)\frac{dy_1}{dt}dt + \int_0^1 x_2(t)\frac{dy_2}{dt}dt\right]$$

$$= \tfrac{1}{4}\left[\int_{\partial\Omega} x_1\,dy_1 + \int_0^1 x_1(t)\frac{dy_2}{dt}dt + \int_0^1 x_2(t)\frac{dy_1}{dt}dt + \int_{\partial\Omega} x_2\,dy_2\right]$$

$$= \tfrac{1}{4}\left[A + \int_0^1 x_1(t)\frac{dy_2}{dt}dt + \int_0^1 x_2(t)\frac{dy_1}{dt}dt + A\right]$$

$$\tfrac{1}{2}A + \tfrac{1}{4}\left[\int_0^1 x_1(t)\frac{dy_2}{dt}dt + \int_0^1 x_2(t)\frac{dy_1}{dt}dt\right]$$

For the remaining integrals, relate these to $\mathbf{u}$ and the circular region R of radius I by writing $x_1(t) = x_2(t) - u_x(t)$ and $x_2(t) = x_1(t) + u_x(t)$. We then get the following:

$$\int_0^1 x_1(t)\frac{dy_2}{dt}dt = \int_0^1 (x_2(t) - u_x(t))\frac{dy_2}{dt}dt = A - \int_0^1 u_x(t)\frac{dy_2}{dt}dt$$

$$\int_0^1 x_2(t)\frac{dy_1}{dt}dt = \int_0^1 (x_1(t) + u_x(t))\frac{dy_1}{dt}dt = A + \int_0^1 u_x(t)\frac{dy_1}{dt}dt$$

Substituting these into the previous result, we have

$$area(\Gamma) = \tfrac{1}{2}A + \tfrac{1}{4}\left[2A - \int_0^1 u_x(t)\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right)dt\right]$$

$$= A - \tfrac{1}{4}\int_0^1 u_x(t)\frac{du_y}{dt}dt = A - \tfrac{1}{4}\int_{\partial R} u_x du_y = A - \tfrac{1}{4}area(R)$$

Finally, since $area(R) = \pi I^2$, we get that $area(\Gamma) = area(\Omega) - \frac{1}{4}\pi I^2$.