PSET 8 SOLS

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1

4. Using the suggested parameterization, the uv-domain D is $1 \le u \le 2$ and $0 \le v \le 2\pi$. Therefore, the surface area is

$$\begin{aligned}\iint_D |X_u(u,v) \times X_v(u,v)|\, dA &= \iint_D |(\cos v, \sin v, 1) \times (-u\sin v, u\cos v, 0)|\, dA \\ &= \int_0^{2\pi}\int_1^2 \sqrt{2}u\, du\, dv \\ &= 3\sqrt{2}\pi.\end{aligned}$$

2

1. (a) If $f(x,y,z) = (x,0,0)$, then $\iint_S f \cdot n\, dS = \iint_D (u,0,0)\cdot(-2u,-2v,1)\, du\, dv = \iint_D -2u^2\, du\, dv = -\frac{\pi}{2}$. (The last integral is done in polar coordinates.) That the answer is negative means that the net flux is across the surface toward the outside.

   (b) If $f(x,y,z) = (0,1,0)$, then $\iint_S f \cdot n\, dS = \iint_D (0,1,0)\cdot(-2u,-2v,1)\, du\, dv = \iint_D -2v\, du\, dv = 0$. The sign of the answer means that the net flux across the surface is zero.

   (c) If $f(x,y,z) = (0,0,z)$, then $\iint_S f \cdot n\, dS = \iint_D (0,0,u^2+v^2)\cdot(-2u,-2v,1)\, du\, dv = \iint_D (u^2+v^2)\, du\, dv = \frac{\pi}{2}$. That the answer is positive means that the net flux is across the surface toward the inside of the bowl.

2. All numerical answers are the same as in the previous exercise.

4. (a) The surface area of a right circular cylinder of radius r and height h is $2\pi rh$. Thus, the surface area is 2π.

   (b) The surface S can be parameterized using cylindrical coordinates as $X(u,v) = (\cos u, \sin u, v)$, $0 \le u \le 2\pi$, $0 \le v \le 1$. Thus, the surface area of S is $\iint_D |X_u \times X_v|\, dA = \int_0^{2\pi}\int_0^1 1\, dv\, du = 2\pi$.

   (c) Using the parameterization from part (b), the flux across S is

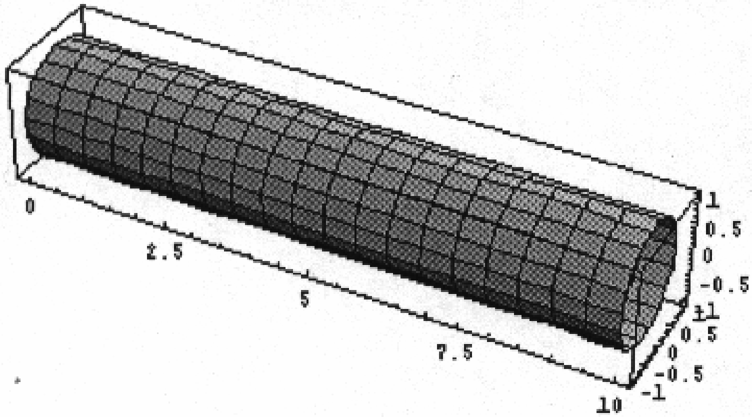
$$\begin{aligned}\iint_S f\cdot n\, dS &= \iint_D f(X(u,v))\cdot|X_u(u,v) \times X_v(u,v)|\, dA \\ &= \iint_D (\cos u, 0, 0)\cdot(\cos u, \sin u, 0)\, dA \\ &= \int_0^{2\pi}\int_0^1 \cos^2 u\, dv\, du = \pi.\end{aligned}$$

   (d) Using the parameterization from part (b), the flux across S is

$$\begin{aligned}\iint_S f\cdot n\, dS &= \iint_D f(X(u,v))\cdot|X_u(u,v) \times X_v(u,v)|\, dA \\ &= \iint_D (0,0,R)\cdot(\cos u, \sin u, 0)\, dA = 0.\end{aligned}$$

Untitled-1 * -STUDENT VERSION-

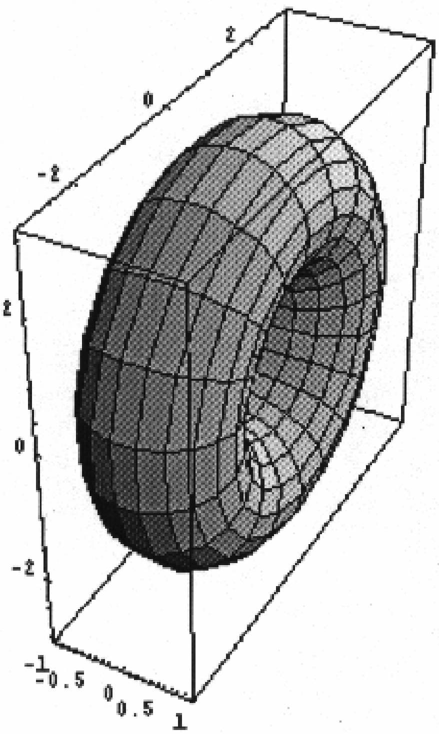
In[13]:= `ParametricPlot3D[{u, Cos[v], Sin[v]}, {u, 0, 10}, {v, 0, 2 Pi}]`



Out[13]= - Graphics3D -

Untitled-1 * -STUDENT VERSION-

In[14]:= `ParametricPlot3D[`
`{Cos[u], (Sin[u] + 2) Cos[v], (Sin[u] + 2) Sin[v]},`
`{u, 0, 2 Pi}, {v, 0, 2 Pi}]`



Out[14]= - Graphics3D -

3) $g(u)$ greater than 0 prevents $(f(u), g(u))$ from crossing x-axis keeping the surface well defined. For a fixed u $(f(u), g(u))$ is a point a distance $g(u)$ from x-axis the surface of rotation sliced at that point is a circle about the x-axis with $g(u)$ as a radius center $(f(u), 0, 0)$

so
$$\left.\begin{aligned} x &= f(u) \\ y &= g(u)\cos v \\ z &= g(u)\sin v \end{aligned}\right\} \quad X(u,v) = (f(u), g(u)\cos v, g(u)\sin v) \quad v \in [0, 2\pi)$$

allowing u to vary these parameters trace the whole surface.

— See MATHEMATICA for SKETCHES.

4) following above.

$C: u \longrightarrow (u, e^u)$ $\qquad e^u > 0 \quad \forall u \in \mathbb{R}$

So

$$X(u,v) = (u, e^u \cos v, e^u \sin v)$$

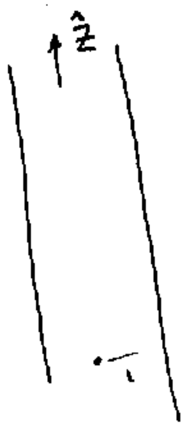
$v \in [0, 2\pi)$ $\quad u \in \mathbb{R}$

5) One parameterization is

$$x = y^2 - z^4 y^4, \quad y = u, \quad z = v$$
$$x = u^2 - v^4 u^4$$

$$X(u,v) = (u^2 - v^4 u^4, u, v) \qquad u, v \in \mathbb{R}$$

6)



$$\sigma(x,y,z) = e^{-|z|}$$

$$Q = \text{charge total} = \iint_{\substack{\text{surface}\\ \text{cyl}}} \sigma \, dS$$

Surface of cylinder is parameterized as

$$X(u,v) = (\cos u, \sin u, v) \qquad u \in [0, 2\pi], \ v \in \mathbb{R}$$

$$X_u = (-\sin u, \cos u, 0), \quad X_v = (0,0,1),$$

$$X_u \times X_v = \cos u \, \hat{i} + \sin u \, \hat{j}, \quad \|X_u \times X_v\| = 1$$

so $A = [0, 2\pi] \times \mathbb{R}$

$$Q = \int_0^{2\pi}\int_{-\infty}^{\infty} \sigma \, \|X_u \times X_v\| \, dv \, du$$

$$= \int_0^{2\pi}\int_{-\infty}^{\infty} e^{-|v|} \, dv \, du$$

$$= 2\pi \int_{-\infty}^{\infty} e^{-|v|} dv = 2\pi \left[\int_0^{\infty} e^{-v} dv + \int_0^{\infty} e^{-v} dv \right]$$

$$= 4\pi \int_0^{\infty} e^{-v} dv = 4\pi$$

7) $\sigma(x,y,z) = z^2$

$$Q = \iint_{\text{Sphere}} \sigma \, dS$$

using spherical coordinate ~~parameter~~ parameterization with $\rho = 1$

$$X(u,v) = (\sin v \cos u, \sin v \sin u, \cos v)$$

~~$X_u = (\sin v \cos$~~

$$X_u = (-\sin v \sin u, \sin v \cos u, 0)$$

$$X_v = (-\cos v \cos u, -\cos v \sin u, -\sin v)$$

$$X_u \times X_v = (-\sin^2 v \cos u, -\sin^2 v \sin u, \sin v \cos v)$$

$$\|X_u \times X_v\| = \sin v \qquad u \in [0, 2\pi] \quad v \in [0, \pi]$$

$$Q = \int_0^{2\pi}\int_0^{\pi} \sigma \sin v \, dv \, du$$

$$\sigma = \cos^2 v$$

$$Q = \int_0^{2\pi}\int_0^{\pi} \cos^2 v \sin v \, dv \, du$$

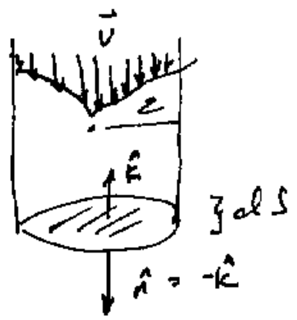
$$Q = 2\pi \int_0^{\pi} \cos^2 v \sin v \, dv = 2\pi \left[-\frac{1}{3}\cos^3 v \right|_0^{\pi}$$

$$Q = 2\pi\left(\frac{2}{3}\right) = \frac{4}{3}\pi$$

8) We can assume that flux is time invariant since $\frac{\partial \vec{V}}{\partial t} = 0$ & $\vec{dA}$ is fixed.

So Volume = $\text{Flux}\left[\frac{in^3}{sec}\right] \cdot 3\,[sec]$

$$\text{Flux} = \iint_S \vec{V} \cdot \hat{n}\, dS$$



$S = \text{Disk}_2(0)$ $[in^2]$

$\vec{V} = (4 - r^2)\hat{k}$ $\left[\frac{in}{sec}\right]$

$$\text{Flux} = \iint_{\text{Disk}} \vec{V} \cdot \hat{k}\, dA = \iint_{\text{Disk}} (r^2 - 4)\hat{k} \cdot -\hat{k}\, dA$$

$$= \int_0^{2\pi}\int_0^2 (4 - r^2)\, r\, dr\, d\theta = 2\pi \int_0^2 (4r - r^3)\, dr$$

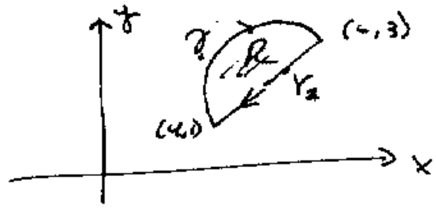
$$= 8\pi \left[\frac{in^3}{sec}\right]$$

Volume = $\cancel{2\pi}$ $24\pi\ [in^3]$

9) Greens thm

$$\oint_\gamma F\cdot dx = \iint_R |(\nabla\times F)|\,dA = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

So



$\gamma = \gamma_1 + \gamma_2$

<u>NOTE:</u> $\gamma_1 \cup \gamma_2$ is a closed curve that traverses boundary of region R <u>clockwise</u>. we must account for this by reversing the sign in Green's theorem.

$$\int_{\gamma_1} + \int_{\gamma_2} = -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_{\gamma_1} = -\iint_R - \int_{\gamma_2}$$

$$|\nabla\times F| = 3 - 1 = 2 = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\int_{\gamma_1} F\cdot dx = -\iint_R 2\,dA - \int_{(6,3)}^{(4,1)} \vec{F}\cdot dx = -\iint_R 2\,dA + \int_{(4,1)}^{(6,3)} \vec{F}\cdot dx$$

Parametrize line segment $+(2,2)t + (4,1) \quad t\in(0,1)$

$$\int_{(4,1)}^{(6,3)} \vec{F}\cdot dx = \int_0^1 [(x+y),(3x-2y)]\cdot([2t+4,\ 2t+1])'\,dt$$

$$= \int_0^1 2\big((2t+4)+(2t+1) + 3(2t+4) - 2(2t+1)\big)\,dt$$

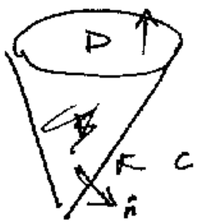
$$= \int_0^1 2(6t+15)\,dt = 6\int_0^1 (2t+5)\,dt = 36$$

$$\iint_R 2\,dA = \int_0^\pi \int_0^{\sqrt{2}} 2r\,dr\,d\theta = 2\pi \int_0^{\sqrt{2}} r\,dr = 2\pi$$

$$\qquad = 2\iint_R dA = 2\cdot \text{area}(R) = 2\cdot \cancel{\#}\tfrac{1}{2}\pi(\sqrt{2})^2 = 2\pi$$

$$\therefore \int_{\gamma_1} \vec{F}\cdot dx = -2\pi + 36$$

10)



Divergence THM

$$\int_V (\nabla\cdot\vec F)\, dV = \int_{\partial V} \vec F\cdot d(\vec{\partial V}) = \int_{Bnd(V)} \vec F\cdot d\vec S$$

$\partial V \leftarrow$ (means Boundary of V)

$$\int_V (\nabla\cdot F)\, dV = \int_D \hat F\cdot d(\vec D) + \boxed{\int_{cone} \vec F\cdot d(\vec{cone}) = \iint_{cone} \vec F\cdot d\vec S}$$

$\uparrow$ means $d\vec S$

$$\text{Flux} = \int_V (\nabla\cdot F)\, dV - \int_D \vec F\cdot d(\vec D)\,, \qquad \underset{=\nabla\cdot F}{\operatorname{div}\vec F} = 0 + 2y + 0 = 2y$$

D is a disk $d\vec S = d(\vec D) = +\hat k\, dA = \hat k\, r\, dr\, d\theta$

V is cone. 2 high 3 across radius.

$\theta = [0, 2\pi]$ $\quad z = z$ $\quad r = \frac{3}{2}z$ $\qquad y = 2r\sin\theta$

$$\text{Flux} = \underbrace{\int_0^{2\pi}\int_0^{2}\int_0^{\frac{3}{2}z} 2r\sin\theta\; r\,dr\,dz\,d\theta}_{\int^{2\pi}\sin\theta\, d\theta = 0} - \int_0^{2\pi}\int_0^3 e^{r^2}\, r\,dr\,d\theta$$

$$= 0 - \int_0^{2\pi}\int_0^3 e^{r^2} r\,dr\,d\theta = -2\pi\int_0^3 e^{r^2} r\,dr$$

$$= -2\pi\left(-\frac{1}{2} + \frac{e^3}{2}\right)$$

$$= \pi(1 - e^3)$$

11) By the divergence Thm

$$\int_D \nabla \cdot F \, dV = \int_{\partial D} F \cdot \hat{n} \, dS = \text{Flux}$$

$\partial D \leftarrow$ (Boundary of D)

$$\nabla \cdot F = y^2 + 2yx$$

in spherical coordinates

$$\nabla \cdot F = (\rho \sin\phi \cos\theta)^2 + 2\rho^2 \sin^2\phi \sin\theta \cos\theta$$

$$D = \{ (\rho, \phi, \theta) \mid \rho \in [0, 3], \phi \in [0, \pi/2], \theta \in [0, \pi/2] \}$$

$$\text{Flux} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \left((\rho^2 \sin^2\phi \cos\theta)(\cos\theta + 2\sin\theta) \right) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^4 \sin^3\phi \cos\theta (\cos\theta + 2\sin\theta) \, d\rho \, d\phi \, d\theta$$