Problem Set #8 Solution Set
Brochem Version

1) $E(X)=\sum x_i p_i = \mu$; $V(X)=\sum (x_i-\mu)^2 p_i = \sigma^2$

In this case $\mu = 3$ and $\sigma^2 = 5$

$E(X-1) = \sum (x_i-1)p_i = \sum x_i p_i - \sum p_i = \sum x_i p_i - 1 = \mu - 1$

$\sum p_i = 1$ because the sum of all probabilities must equal to 1 by definition.

$V(X-1) = \sum ((x_i-1)-(\mu-1))^2 p_i = \sum (x_i-\mu)^2 p_i = \sigma^2$

$E(2X) = \sum 2x_i p_i = 2\sum x_i p_i = 2\mu$

$V(2X) = \sum (2x_i - 2\mu)^2 p_i = \sum 4(x_i-\mu)^2 p_i = 4\sigma^2$

Thus; $\boxed{E(X-1) = \mu - 1 = 2}$ $\boxed{V(X-1) = \sigma^2 = 5}$

$\boxed{E(2X) = 2\mu = 6}$ $\boxed{V(2X) = 4\sigma^2 = 20}$

2)

$E(X^2-1) = \sum (x_i^2-1)p_i = \sum x_i^2 p_i - \sum p_i = \sum x_i^2 p_i - 1$

but $\sum x_i^2 p_i = ?$ well $(x_i-\mu)^2 = x_i^2 - 2\mu x_i + \mu^2$

$\Rightarrow x_i^2 = (x_i-\mu)^2 + 2\mu x_i - \mu^2$

$\Rightarrow \sum x_i^2 p_i = \sum (x_i-\mu)^2 p_i + 2\mu \sum x_i p_i - \mu^2 \sum p_i$

$= \sigma^2 + 2\mu^2 - \mu^2 = \sigma^2 + \mu^2$

$\Rightarrow E(X^2-1) = \sigma^2 + \mu^2 - 1 = 5 + 9 - 1 = \boxed{13}$

3)

a) The probability of at least 2 errors is "not" the probability of 0 or 1 errors:

$P(0\text{ errors}) = \frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10} = \frac{6561}{10{,}000} = 0.6561$

$P(1\text{ error}) = \frac{1}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot$ (# of ways of getting 1 error)

$= \frac{729}{10{,}000}\left(\frac{4!}{3!1!}\right) = 0.2916$

$P(0\text{ or }1\text{ error}) = 0.6561 + 0.2916 = 0.9477$

So the $P(\text{at least 2 errors}) = 1 - 0.9477 = \boxed{0.0523}$

b) $P(\text{at most 2 errors}) = 1 - P(3\text{ or }4\text{ errors})$

$P(3\text{ errors}) = \left(\frac{1}{10}\right)^3 \cdot \frac{9}{10} \cdot \frac{4}{3!1!} = 0.0036$

$P(4\text{ errors}) = \left(\frac{1}{10}\right)^4 = 0.0001$

$P(3\text{ or }4\text{ errors}) = 0.0001 + 0.0036 = 0.0037$

$P(\text{at most }2) = 1 - 0.0037 = \boxed{0.9963}$

$P(A\cap B) =$ clearly $P(2\text{ errors}) = \left(\frac{1}{10}\right)^2 \cdot \left(\frac{9}{10}\right)^2 \cdot \frac{4!}{2!2!} = 0.0486$

$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0.0486}{0.0523} = 0.929254 \neq P(B)$

c) B and A are not independent since

$P(B|A) \neq P(B)$

4) We have 50 cases to choose from

5 spots 50 for first 49 for second ...

$\underline{50}\cdot\underline{49}\cdot\underline{48}\cdot\underline{47}\cdot\underline{46} =$

$\boxed{254{,}251{,}200}$ this is the final result because order matters

5) If order doesn't matter then we have to divide by the number of permutations we can make of each set of 5; number of selections $= \frac{254251200}{5!} = \boxed{2118760}$

6)

$P(X=k) = \left(\frac{1}{e}\right)\left(\frac{1}{k!}\right)$; $P(Y=k) = \left(\frac{1}{e^2}\right)\left(\frac{2^k}{k!}\right)$

It seems clear that X and Y are indep; the amount of bacteria on plate X has nothing to do w/ that on plate Y

Hence $P(X+Y)=0 = P(X=0)\cap P(Y=0) = P(X=0)\cdot P(Y=0)$

$= \left(\frac{1}{e}\right)\left(\frac{1}{e^2}\right) = \boxed{\frac{1}{e^3}}$

b) $P(X+Y)=1 = [P(X=0)P(Y=1)] + [P(X=1)P(Y=0)]$

$= \left[\left(\frac{1}{e}\right)\cdot\left(\frac{2}{e^2}\right)\right] + \left[\left(\frac{1}{e}\right)\left(\frac{1}{e^2}\right)\right] = \boxed{\frac{3}{e^3}}$

c) We have 8 cases to sum here

$X=0\ \left(\frac{1}{e}\right)$ $Y=8\ \left(\frac{1}{e}\right)\left(\frac{1}{e^2}\right)\left(\frac{2^8}{8!}\right)$

$X=1$ $=7\ \left(\frac{1}{e^3}\right)\left(\frac{2^7}{7!}\right)$

$X=2$ $=6$ $\left(\frac{1}{2!}\right)\left(\frac{2^6}{6!}\right)$

$X=3$ $=5$

$X=4$ $=4$

$X=5$ $=3$

$X=6$ $=2$

$X=7$ $=1$

$X=8$ $=0$

$$\left(\frac{1}{e^3}\right)\left[\left(\frac{2^8}{8!}\right)+\left(\frac{2^7}{7!}\right)+\left(\frac{1}{2!}\right)\left(\frac{2^6}{6!}\right)+\left(\frac{1}{3!}\right)\left(\frac{2^5}{5!}\right)+\left(\frac{1}{4!}\right)\left(\frac{2^4}{4!}\right)+\left(\frac{1}{5!}\right)\left(\frac{2^3}{3!}\right)+\left(\frac{1}{6!}\right)\left(\frac{2^2}{2!}\right)+\left(\frac{1}{7!}\right)\left(\frac{2^1}{1}\right)+\left(\frac{1}{8!}\right)\right] = \left(\frac{1}{e^3}\right)\left[\frac{2^8+1}{8!}+\frac{2^7+2}{7!}+\left(- - - - \right)\right]$$

$$= \frac{1}{e^3}\left[\frac{729}{4480}\right) = \boxed{\frac{729}{4480e^3}}$$

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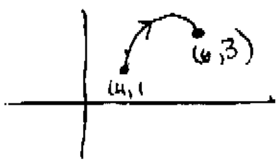
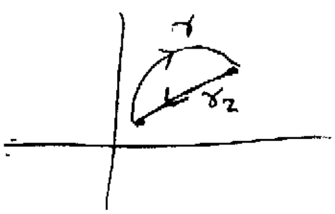


It would be really hard to parametrize the semi-circle, but it would be really easy to just use Green's theorem;
However, to do so we need a <u>closed</u> loop
so lets close it by using the line between
$(6,3)$ and $(4,1) = \gamma_2 =$
$(6,3) + t(-2,-2)$; $0 \le t \le 1$



Interestingly enough, $Q_x - P_y = 3-1 = 2$

hence $\oint_{\gamma \cap \gamma_2} F \cdot \vec{dt} = -\iint_{\text{half circle}} 2\,dx\,dy = \underset{\substack{\uparrow \\ \text{negative because boundary} \\ \text{is oriented } \underline{\text{clockwise}}.}}{-2}$ (Area of half circle)

the radius of the half circle is $\frac{1}{2}$ distance between the points $= \frac{1}{2}\sqrt{2^2+2^2} = \sqrt{2}$

So area of half circle $= \frac{\pi r^2}{2} = \frac{\pi 2}{2} = \pi$

Therefore $-2\pi = \oint_{\gamma\cap\gamma_2} F \cdot dt = \oint_{\gamma} F \cdot dt + \oint_{\gamma_2} F \cdot dt$

$\oint_{\gamma_2} F \cdot \vec{dt} = \int_0^1 F(\gamma_2(t)) \cdot \gamma_2'(t)\,dt$

$F(\gamma_2(t)) = (9-4t,\ 12-2t)$

$\gamma_2'(t) = (-2,-2)$

$\oint_0^1 F(\gamma_2(t)) \cdot \gamma_2'(t)\,dt = \oint_0^1 (-18+8t-24+4t)\,dt = \oint_0^1 (12t-42)\,dt$

$= \left[6t^2 - 42t\right]_0^1 = -36$

$\int_{\gamma} F \cdot dt = -2\pi - \oint_{\gamma_2} F \cdot dt =$

$= \boxed{36 - 2\pi}$