Math 21a - Spring 2001 - Problem Set #9 (Regular/Physics sections)

Dates covered: April 23 - May 2
Text sections: 5.6, 5.7, and supplements
Key topics: Divergence of a vector field and the Divergence Theorem; curl of a vector field and Stokes'
Theorem; five versions of the Fundamental Theorems of Calculus; other integration problems.

(1) Ostebee-Zorn problems: 5.7/4,6,8,10

(2) Compute the volume of the tetrahedron V (four sided prism) in $\mathbf{R}^3$ given by $x \geq 0$, $y \geq 0$, $z \geq 0$ and
$x + y + z \leq 1$. (Thus, the boundary of V has four triangular faces, one each in the planes $x = 0$, $y = 0$,
$z = 0$ and $x + y + z = 1$.) Do the computation first as an iterated integral, and then via the Divergence
Theorem as a flux integral through the boundary of V of the electric field $\mathbf{E} = (x, 0, 0)$.

(3) In each case, compute the line integral of the given vector field $\mathbf{F}$ about the boundary curve, γ, of the
triangle in the plane $x + y + z = 1$ where x, y and z are all positive. In each case, traverse γ clockwise as
viewed looking down from the positive z-axis. Do this by first parametrizing each of the three
segments of γ. Then, do the calculation via Stokes' Theorem.:
   (a) $\mathbf{F} = (z, x, y)$.
   (b) $\mathbf{F} = (x^2, y, z)$.
   (c) $\mathbf{F} = (xy, z, 0)$.

(4) In each case below, either write down a vector field on $\mathbf{R}^3$ with the desired properties, or else explain
why no such vector field exists.
   (a) The curl is (1, 1, 0) and the flux is zero through any surface in a plane where z is constant.
   (b) The curl is (1, 1, 0) and the line integral is zero around all loops in the $x = 0$ plane.
   (c) The divergence is 2 and the line integral is zero around all closed loops.
   (d) The divergence is 2 and the flux through any closed surface is zero.

(5) Let $\mathbf{F}$ be a vector field on $\mathbf{R}^3$ and let g be a function on $\mathbf{R}^3$. Prove the following "product rules":
   a) $\text{div}(g\,\mathbf{F}) = g\,\text{div}(\mathbf{F}) + \nabla g \cdot \mathbf{F}$.
   b) $\text{curl}(g\,\mathbf{F}) = g\,\text{curl}(\mathbf{F}) + \nabla g \times \mathbf{F}$.

(6) Suppose that $\mathbf{E}$ is an electric field (a vector field) on $\mathbf{R}^3$ and $\|\mathbf{E}\| \leq 1$ at all points inside the ball B
where $x^2 + y^2 + z^2 \leq 1$. Explain why $-4\pi \leq \iiint_B \text{div}(\mathbf{E})\,dV \leq 4\pi$. (Hint: Use the Divergence Theorem
to rewrite this integral and then think about the size of the resulting integrand.)

(7) Let $\mathbf{A}$ be a vector potential for the vector field $\mathbf{F}$, i.e. a vector field $\mathbf{A}$ such that curl $\mathbf{A} = \mathbf{F}$. Further
suppose that $\|\mathbf{A}\| \leq 1$ at all points in the disk D in the $z = 0$ plane where $x^2 + y^2 \leq 1$. Use $\mathbf{n} = (0, 0, 1)$
for D's normal vector. Explain why the flux of $\mathbf{F}$ through D has absolute value no greater than 2π.
That is, why is it true that $-2\pi \leq \iint_D \mathbf{F} \cdot \mathbf{n}\,dS \leq 2\pi$?

(8) For each angle θ between 0 and 2π, let D_θ denote the disk in $\mathbf{R}^3$ whose radius is 1, center is the origin
in $\mathbf{R}^3$ and which lies in the plane $\cos(\theta)\,x + \sin(\theta)\,z = 0$. For each θ, use $I(\theta)$ to denote the absolute
value of the path integral over the boundary of D_θ of the vector potential $\mathbf{A} = (0, 0, y)$. What is the
maximum value of $I(\theta)$ and what are the angles θ which have this value? (Hint: The problem is easier
if you use Stokes' Theorem and think about how the size of $\text{curl}(\mathbf{A}) \cdot \mathbf{n}$ depends on the angle θ.)

(9) (a) Find the average value of y for the vertical line segment (in $\mathbf{R}^2$) from the origin to the point $(0, a)$. [The answer should be obvious. It's just a warm-up.]

   (b) Find the average value of y over the semicircle of radius a (in $\mathbf{R}^2$) given by $x^2 + y^2 = a^2$, $y \geq 0$.

   [The average value of a function f over a curve γ is given by $\bar{f} = \dfrac{\int_\gamma f\, ds}{\text{length}(\gamma)}$ where ds represents the element of arclength along the curve. If γ is parametrized by $\mathbf{x}(t)$, then $ds = \|\mathbf{v}\|dt$.]

   (c) Find the average value of z over the hemisphere (in $\mathbf{R}^3$) given by $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

   (d) Find the average value of z over the upper half of the solid ball given by $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$.

(10) Let $\mathbf{F} = (x + xz, y - yz, z^2)$. Here are two surfaces in $\mathbf{R}^3$ with the same boundary:

$$\text{A:}\quad z = \sqrt{1 - x^2 - y^2}\quad \text{with } x^2 + y^2 \leq 1.$$
$$\text{B:}\quad z = 2\sqrt{1 - x^2 - y^2}\quad \text{with } x^2 + y^2 \leq 1.$$

   (a) Determine which of these two surfaces has the lowest flux of $\mathbf{F}$ and which has the highest. Ties are allowed. Use the normal which has positive dot product with $(0, 0, 1)$. Justify your answer.

   (b) Determine which of these two surfaces has the lowest flux of $\text{curl}(\mathbf{F})$ and which has the highest. Ties are allowed here too. Use the same normal as in part (a). Justify your answer.

   (c) Compute the flux of $\text{curl}(\mathbf{F})$ through the surface where $z = 1 - x^2 - y^2$ with $x^2 + y^2 \leq 1$.