5.7   Back to fundamentals: Stokes' theorem and the divergence theorem

1. (a) If S is the upper half of the sphere, then the boundary curve γ is the unit circle. We can parametrize it by
    $x = \cos t$, $y = \sin t$, $z = 0$, and the line integral becomes $\int_\gamma x\,dx + y\,dy + z\,dz = \int_0^{2\pi} 0\,dt = 0$.

    (b) S and γ are as in the preceding part. Now the line integral becomes $\int_\gamma -y\,dx + x\,dy + z\,dz = \int_0^{2\pi} 1\,dt = 2\pi$.

    (c) If S is the part of the paraboloid $z = x^2 + y^2$ with $z \le 1$, then the boundary curve γ is the unit circle, as
    in the preceding parts. Now the line integral becomes $\int_\gamma y\,dx + z\,dy + x\,dz = \int_0^{2\pi} 0 - \sin^2 t\,dt = -\pi$.

2. Using cylindrical coordinates, $\iiint_V \operatorname{div} f\,dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r\,dz\,dr\,d\theta = \dfrac{\pi}{2}$.

    Using polar coordinates, $\iint_{S_1} f \cdot n\,dS = \iint_D (u, v, u^2 + v^2) \cdot (2u, 2v, -1)\,du\,dv = \iint_D (u^2 + v^2)\,du\,dv$

    $= \int_0^{2\pi} \int_0^1 r^3\,dr\,d\theta = \dfrac{\pi}{2}$.

    Using polar coordinates, $\iint_{S_2} f \cdot n\,dS = \iint_D (u, v, 1) \cdot (0, 0, 1)\,du\,dv = \iint_D du\,dv = \int_0^{2\pi} \int_0^1 r\,dr\,d\theta = \pi$.

3. (a) Here $\operatorname{div} f = 1$, so the the volume integral $\iiint_V \operatorname{div} f\,dV$ is just the volume of V, or $\pi/2$. (The volume is
    simplest in spherical coordinates.)

    (b) With parametrizations as in Example 2, the surface integral has two parts, over S_1 and S_2. The flux integral
    over S_1 becomes $\iint_D (u, 0, 0) \cdot (2u, 2v, -1)\,du\,dv = \iint_D 2u^2\,du\,dv$, where D is the unit disk. The value
    of this integral is $\pi/2$.
    The flux integral over S_2 becomes $\iint_D (u, 0, 0) \cdot (0, 0, 1)\,du\,dv = 0$.
    The total flux integral, therefore, is $\pi/2$.

4. (a) The divergence of the given vector field is 3, so the divergence theorem says here that the flux integral
    across S is the volume integral of the function 3 on the solid V.

    (b) We get $\iiint_V \operatorname{div} f\,dV = \iiint_V 3\,dV = 3$ volume of $V = 3$.

    (c) The flux integral has 6 parts; the answers add up to 3. Notice that along each face, a suitable normal vector
    is i, j, or k.

5. The divergence of any curl field is zero; therefore, by the divergence theorem,

$$\iint_S f \cdot n\,dS = \iiint_V \operatorname{div} \nabla h\,dV = 0.$$

6. (a) Since $\operatorname{div} f = 6$, $\iint_S f \cdot n\,dS = 6 \cdot 4\pi/3 = 8\pi$.

    (b) Since $\operatorname{div} f = 1 + 2y$, $\iint_S f \cdot n\,dS = \int_0^{2\pi} \int_0^\pi \int_0^1 (1 + 2\rho \sin\phi \sin\theta)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \dfrac{4\pi}{3}$.

    (c) Since $\operatorname{div} f = 2y$, $\iint_S f \cdot n\,dS = \int_0^{2\pi} \int_0^\pi \int_0^1 2\rho^3 \sin^2\phi \sin\theta\,d\rho\,d\phi\,d\theta = 0$

7. (a) Let $X(t) = (\cos t, \sin t, 0)$. Then, $\int_\gamma f \cdot dX = \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)\,dt = 2\pi$.

    (b) Since γ is the boundary of the upper half of the unit sphere, Stokes' theorem says that $\iint_S (\operatorname{curl} f) \cdot n\,dS = \int_\gamma f \cdot dX = 2\pi$. This result can be checked by evaluating the surface integral directly: Since $\operatorname{curl} f = (0, 0, 2)$, $\iint_S (\operatorname{curl} f) \cdot n\,dS = \iint_A (0, 0, 2) \cdot (-g_u, -g_v, 1)\,dA = 2 \iint_D dA = 2\pi$, where $g(u, v) = \sqrt{1 - u^2 - v^2}$ and D is the unit disk $x^2 + y^2 \le 1$.

(c) Since γ is the boundary of S, Stokes' theorem says that $\iint_S (\text{curl } f) \cdot n \, dS = \int_\gamma f \cdot dX = 2\pi$. As in part (b), this result can be checked by evaluating the surface integral directly: $\iint_S (\text{curl } f) \cdot n \, dS = \iint_D (0, 0, 2) \cdot (-2u, -2v, 1) \, dA = 2 \iint_D dA = 2\pi$ since D is the unit disk $x^2 + y^2 \le 1$.

8. (a) curl $f = (1, 1, 1)$

    (b) div $f = -3$

    (c) By the divergence theorem, $\displaystyle\iint_T f \cdot n \, dS = -3 \cdot \frac{4\pi}{3} 2^3 = -32\pi < 0$.

9. (a) The surface S can be described using the parametric representation $X(u, v) = (u, v, u^2 + v^2)$. Therefore, the vector $n = X_u(u, v) \times X_v(u, v) = (-2u, -2v, 1)$ is orthogonal to S at the point $(u, v, X(u, v))$. It follows that the vector $(-2, -4, 1)$ is normal to S at the point $(1, 2, 5)$.

    (b) Let $n = (-2, -4, 1)$ and $X_0 = (1, 2, 5)$. An equation of the plane tangent to S at the point $(1, 2, 5)$ is then $n \cdot (X - X_0) = 0$. Written in scalar form, this equation is $2x + 4y - z = 5$.

    (c) The area of S is $\displaystyle\int_1^2 \int_2^3 |X_u(u, v) \times X_v(u, v)| \, dv \, du = \int_1^2 \int_2^3 \sqrt{1 + 4u^2 + 4v^2} \, dv \, du$.

    (d) The area of the portion of the tangent plane that lies above R is $|X_u(1, 2) \times X_v(1, 2)| = |(-2, -4, 1)| = \sqrt{21}$.

10. Let S denote the unit ball. By the divergence theorem, $\displaystyle\iiint_S \text{div } f \, dV = \iint_{\partial S} f \cdot n \, dS$. Now, since div $f = a$ in all of $\mathbb{R}^3$, $\displaystyle\iiint_S \text{div } f \, dV = \frac{4\pi a}{3}$. Also, since the unit sphere can be parameterized as $X(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$ for $0 \le u \le \pi$ and $0 \le v \le 2\pi$,

$$\iint_{\partial S} f \cdot n \, dS = \iint_D f(X(u, v)) \cdot (X_u(u, v) \times X_v(u, v)) \, dA$$
$$= \int_0^{2\pi} \int_0^\pi (0, 0, \cos^3 u) \cdot (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u) \, du \, dv$$
$$= \int_0^{2\pi} \int_0^\pi \sin u \cos^4 u \, du \, dv = \frac{4\pi}{5}.$$

Therefore, $a = 3/5$.

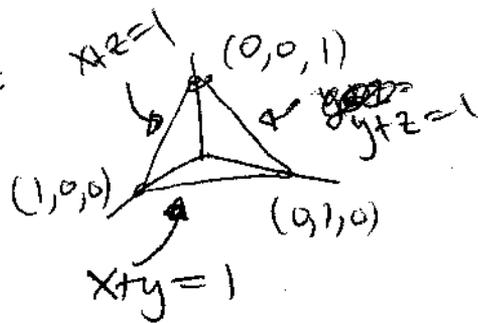11. Let Π be a plane, N a vector normal to Π, γ a closed curve lying in Π, and R the region in Π bounded by γ. Since R is a plane region and curl f is a constant vector,

$$\iint_R (\text{curl } f) \cdot n \, dS = ((\text{curl } f) \cdot N) \iint_R dA.$$

Now, Stokes' theorem implies that $\displaystyle\iint_R (\text{curl } f) \cdot n \, dS = \oint_\gamma f \cdot dX$. Therefore, if $(\text{curl } f) \cdot N = 0$, then $\displaystyle\oint_\gamma f \cdot dX = 0$. It follows that if N is a vector such that $(1, 2, 5) \cdot N = 0$ (e.g., $N = (-1, -2, 1)$), then the plane described by the equation $N \cdot (x, y, z) = 0$ has the desired properties (e.g., $x + 2y = z$).

〰️

2. as an iterated integral:

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dz\,dy\,dx$$

(diagram of tetrahedron with vertices $(0,0,1)$, $(1,0,0)$, $(0,1,0)$; labels $x+z=1$, $y+z=1$, $x+y=1$)

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y)\,dy\,dx = \int_{x=0}^{1} \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \left. \frac{(1-x)^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} + \frac{1}{2}\frac{(1-x)^3}{3} \right]_{0}^{1}$$

$$= -\frac{1}{2} + \frac{1}{3} + \frac{1}{2} - \frac{1}{6}$$

$$= \boxed{\frac{1}{6}}$$

as a flux integral:
The divergence thm says:

$$\iiint_V \text{div } E\, dV = \iiint_V dV = \iint_S E \cdot n\, dS$$

Let T_1 be the triangular face of V with $x=0$

$\quad T_2$ " " " " " with $y=0$

$\quad T_3$ " " " " " with $z=0$

$\quad T_4$ " " " " " $x+y+z=1$.

✦ Note ① $\iint_{T_1} E \cdot dS = 0$ since E is 0 on T_1.

Let n_2, normal to T_2 be $n_2 = (0,-1,0)$

Let n_3, normal to T_3 be $n_3 = (0,0,-1)$

Let n_4, normal to T_4 be $n_4 = \frac{(1,1,1)}{\sqrt{3}}$

✦ Note n_2, n_3, n_4 are all <u>UNIT</u>, outward pointing vectors.

2(cont). It is clear that the area magnification factor is 1 for T_2, T_3, while for T_4 we have $X(u,v) = (u, v, 1-u-v)$ so $X_u = (1, 0, -1)$, $X_v = (0, 1, -1)$ so that $\|X_u \times X_v\| = \sqrt{3}$

$$Flux = \int_{x=0}^{1} \int_{z=0}^{1-x} (X, 0, 0) \cdot (0, -1, 0) \, dz \, dx$$

$$+ \int_{x=0}^{1} \int_{y=0}^{1-x} (X, 0, 0) \cdot (0, 0, -1) \, dy \, dx$$

$$\cancel{\text{///////}} + \int_{z=0}^{1} \int_{y=0}^{1-z} (1-y-z, 0, 0) \cdot (1, 1, 1) \, dy \, dz$$

$$= \int_{z=0}^{1} \cancel{\text{///}} \left((1-z) + (1-z)^2 z - \frac{(1-z)^3}{2} \right) dz = \boxed{\frac{1}{6}}$$

$$\left(\text{Of carse, we cald have done this in kindergarten. } V = \frac{1}{3} bh = \frac{1}{3} \left(\frac{1}{2} \right) 1 = \frac{1}{6} \right).$$
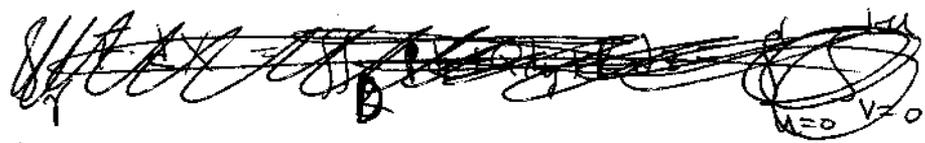
3) Let's first parameterize γ:

γ consists of three lines, the line $\gamma_1: x+y=1$, the line $\gamma_2: x+z=1$, the line $\gamma_3: y+z=1$

We have the following parameterizations:

$$\gamma_1: (t, 1-t, 0) \qquad \gamma_2: \overset{(1-t,\,0,\,t)}{\cancel{\text{(t,t,t)}}} \qquad \gamma_3: (0, t, 1-t)$$
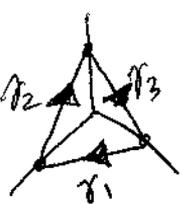
LINE a)
$$\int_\gamma F \cdot dx = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^1 (0, t, 1-t) \cdot (1, -1, 0)\, dt$$

$$+ \int_0^1 (t, 1-t, 0) \cdot (-1, 0, 1)\, dt + \int_0^1 (1-t, 0, t) \cdot (0, 1, -1)\, dt$$

$$= \int_0^1 -t - t - t\, dt = -\frac{3t^2}{2}\Big]_0^1 = \boxed{-\frac{3}{2}}$$

STOKES

$$\int_\gamma F \cdot dx = \iint_D (\text{curl} f) \cdot (1, 1, 1)\, dx\, dy\, dz$$

$$\text{curl} f = (1, 1, 1)$$

$$\int_\gamma F \cdot dx = \int_{z=0}^1 \int_{y=0}^{1-z} 3\, dy\, dz = \int_{z=0}^1 3(1-z)\, dz = -\frac{3}{2}(1-z)^2\Big]_0^1$$

$$= \boxed{\frac{3}{2}}$$

note the sign difference because we parameterized γ clockwise and **NOT** counterclockwise.

<u>LINE</u> b) $\int_{\partial} F \, dX = \int_0^1 (t^2, 1-t, 0) \cdot (1, -1, 0) \, dt + \int_0^1 ((1-t)^2, 0, t) \cdot (-1, 0, 1) \, dt$

$$+ \int_0^1 (0, t, 1-t) \cdot (0, 1, -1) \, dt$$

$$= \int_0^1 t^2 - (1-t) - (1-t)^2 + t + t - (1-t) \, dt$$

$$= \int_0^1 2t - 2 + 2t + 2t - 1 \, dt$$

$$= \int_0^1 6t - 3 \, dt = 3t^2 - 3t \Big]_0^1 = \boxed{0}$$

<u>STOKES</u> $\text{curl} \, F = (0, 0, 0)$ so $\iint_D \text{curl} F \cdot n \, dS = \boxed{0}$

<u>LINE</u> c) $\int_{\partial} F \, dV = \int_0^1 (t(1-t), 0, 0)(1, -1, 0) \, dt + \int_0^1 (0, t, 0) \cdot (-1, 0, 1) \, dt$

$$+ \int_0^1 (0, 1-t, 0) \cdot (0, 1, -1) \, dt$$

$$= \int_0^1 t - t^2 + 1 - t \, dt = t - \frac{t^3}{3} \Big]_0^1 = \boxed{\frac{2}{3}}$$

<u>STOKES</u> $\text{curl} F = (-1, 0, -x)$

$$\iint_D \text{curl} F \cdot n \, dS = \int_{z=0}^1 \int_{y=0}^{1-z} (-1, 0, -(1-y-z)) \cdot (1, 1, 1) \, dy \, dz$$

$$= \int_{z=0}^1 \int_{y=0}^{1-z} (-2 - y - z) \, dy \, dz$$

$$= - \int_{z=0}^1 2(1-z) - \frac{(1-z)^2}{2} - z(1-z) \, dz$$

$$= (1-z)^2 - \frac{(1-z)^3}{6} + \frac{z^2}{2} - \frac{z^3}{3} \Big]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} - 1 + \frac{1}{6} = \frac{1}{3} - 1 = \boxed{-\frac{2}{3}}$$

again, the sign difference comes from the
CW instead of CCW parameterization of γ_0

4a) Let $F = (P, Q, 0)$ then
since $n = (0,0,1)$ is a good
unit normal vector for the plane $z=c$,
we have $F \cdot n = 0$. We want
$$\text{curl } F = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$
$$= (-Q_z, P_z, Q_x - P_y) = (1,1,0)$$
So set $Q = -z$, $P = z$. $\boxed{F = (z, -z, 0)}$ works

b) If the line integral around all closed loops in
the $x=0$ plane is 0, we have
$$\iint_S \text{curl } f \cdot (1,0,0) \, dS = 0 = \iint_S \, dS$$

eg $\text{Area}(S) = 0$ for all regions S bounded by
a closed curve. This is absurd, so $\boxed{\text{NO SUCH VECTOR FIELD EXISTS}}$

c) If the line integral is 0 around all closed
loops, $F = \nabla f$ for some function f.
Then $\text{div } F = f_{xx} + f_{yy} + f_{zz} = 2$. Set $f(x,y,z) = z^2$
Then $\boxed{F = (0,0,2z)}$ and $\text{div } F = 2$.

d) Since
$$\iiint_V \text{div} F \, dV = \iint_S F \cdot n \, dS = 0$$
we have $\iiint_V 2 \, dV = 2 \text{Vol } V = 0$
for all regions V bounded by a closed surface.
This is absurd, so $\boxed{\text{NO SUCH VECTOR FIELD EXISTS}}$

5) $F: \mathbb{R}^3 \to \mathbb{R}^3$, $g: \mathbb{R}^3 \to \mathbb{R}$.

<u>NOTE</u> gF denotes scalar multiplication of the vector F by the scalar g ✓!!

a) Let $F = (P, Q, R)$

$\text{div}(gP, gQ, gR) = (gP)_x + (gQ)_y + (gR)_z$

$= g_x P + g P_x + g_y Q + g Q_y + g_z R + g R_z$

$= (g_x, g_y, g_z) \cdot (P, Q, R) + g(P_x, Q_y, R_z)$

$= g \, \text{div} F + \nabla g \cdot F$

b) $\text{curl}(gF) = \text{curl}(gP, gQ, gR)$

$= \left((gR)_y - (gQ)_z, \ (gP)_z - (gR)_x, \ (gQ)_x - (gP)_y \right)$

$= \left(g_y R + g R_y - g_z Q - g Q_z, \ g_z P + g P_z - g_x R - g R_x, \ g_x Q + g Q_x - g_y P - g P_y \right)$

$= g(R_y - Q_z, \ P_z - R_x, \ Q_x - P_y) \ \cancel{\text{curl} F}$

$+ \left(g_y R - g_z Q, \ g_z P - g_x R, \ g_x Q - g_y P \right)$

<u>Note</u> $\nabla g \times F = \det \begin{vmatrix} i & j & k \\ g_x & g_y & g_z \\ P & Q & R \end{vmatrix} = (g_y R - g_z Q, \ g_z P - g_x R, \ g_x Q - g_y P)$

So $\text{curl}(gF) = g \, \text{curl}(F) + \nabla g \times F$. ▨

6. $\left| \iiint_B \text{div} \, E \, dV \right| = \left| \iint_S E \cdot n \, dS \right| \le \iint_S |E \cdot n| \, (dS)$

Since $\|E\| \le 1$, $|E \cdot n| \le 1$ so that the above is $\le \iint_S dS = $ Surface area of the ~~sphere~~ ball of radius $1 = 4\pi$. Hence,

$\left| \left(\iiint_B \text{div} \, E \, dV \right) \right| \le 4\pi$ which gives the proposed inequality.

7. Since $\iint_S (\text{curl} \, F) \cdot n \, dS = \int_\gamma F \cdot dx$

we have $\iint_S F \cdot n \, dS = \int_\gamma A \cdot dx$

Hence, $\left| \iint_D F \cdot n \, dS \right| = \left| \int_\gamma A \cdot dx \right| \le \int_\gamma |A \cdot dx|$

Since $\|A\| \le 1$, the above is $\le \int_\gamma |dx| = $ length of $\gamma = $ circumference of circle $D = 2\pi$. Thus,

$\left| \iint_D F \cdot n \, dS \right| \le 2\pi$.

8) The plane $\cos\theta x + \sin\theta z = 0$ has unit normal vector $n = (\cos\theta, 0, \sin\theta)$. Since $A = (0,0,y)$, $\operatorname{curl} A = (1,0,0)$. So that $\operatorname{curl} A \cdot n = \cos\theta$. Since

$$I(\theta) = \left| \oint_{\partial D_\theta} A \cdot dx \right| = \left| \iint_{D_\theta} \operatorname{curl} A \cdot n \, dS \right|, \text{ we see}$$

that the left hand side is maximized when $\cos\theta$ is 1 or -1, eg when $\theta = 0, \pi$ (since $0 \le \theta < 2\pi$). By Stokes theorem,

~~crossed out~~

$$I(0) = I(\pi) = \left| \iint_{z^2 + y^2 = 1} 1 \, dS \right| = 2\pi$$

(area of circle of radius 1).

9) a) Ave val $= \dfrac{\int_0^a y \, dy}{\int_0^a dy} = \dfrac{\frac{a^2}{2}}{a} = \boxed{\frac{a}{2}}$

b) $\gamma(t) = (\cos t, \sin t)$

Ave val $= \dfrac{\int_0^\pi (\sin t) \cdot 1 \, dt}{\pi}$

(since $\|\gamma'(t)\| = 1$) $= \boxed{\dfrac{2}{\pi}}$

c) ~~Area~~ ~~~~~~~~~~

Parameterize the hemisphere by
$X(u,v) = (a \sin v \cos u, a \sin v \sin u, a \cos v)$
(in spherical coordinates). We have
$X_u = (-a \sin v \sin u, a \sin v \cos u, 0)$
$X_v = (a \cos v \cos u, a \cos v \sin u, -a \sin v)$
$X_u \times X_v = (-a^2 \sin^2 v \cos u, -a^2 \sin^2 v \sin u, -a^2 \sin v \cos v)$

$\|X_u \times X_v\| = a^2 \sin v$

In terms of our parameterization $z = a \cos v$.

we have $\quad$ Ave Val $= \dfrac{\displaystyle\int_{v=0}^{\pi/2} \int_{u=0}^{2\pi} (a \cos v)(a^2 \sin v) \, du \, dv}{\frac{1}{2} \cdot 4 \pi a^2}$

$= \dfrac{a^3}{2\pi a^2} \, 2\pi \, \left[\dfrac{\sin^2 v}{2} \right]_0^{\pi/2}$

$\boxed{= \dfrac{a}{2}}$

d) $\quad$ Ave Val $= \dfrac{\displaystyle\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} (\rho \cos\phi) \, \rho^2 \sin\phi \,) \, d\rho \, d\phi \, d\theta}{\frac{1}{2} \cdot \frac{4}{3} \pi a^3}$

$= \dfrac{2\pi \left(\frac{a^4}{4} \right) \left[\frac{\sin^2\phi}{2} \right]_0^{\pi/2}}{\frac{2}{3} \pi a^3} = \dfrac{a^4/4}{\frac{2}{3} a^3} \quad \boxed{\dfrac{3}{8} a}$

10)a) Since $\iiint_V \operatorname{div} f \, dV = \iint_S f \cdot n \, ds$

and since $\operatorname{div} F = \cancel{\text{\textit{scribbled}}} \; 1 + z + 1 - z + 2z$
$= 2(1+z)$ we have

$2\iiint_V (1+z) \, dV = $ Flux of F through surface
$\qquad\qquad\qquad\qquad$ + flux of F through bandary

both A, B have the same flux through the bandary (since the bandaries are the same)
So the flux through the surface is least
for surface $\boxed{A}$ since $2\iiint_{V_1} (1 + \sqrt{1-x^2 y^2}) \, dV$

is $\underline{\text{clearly}}$ less than $2\iiint_{V_2} (1 + 2\sqrt{1-x^2 y^2}) \, dV$.

b) Since both surfaces have the same banding curve, and since

$$\iint_S \operatorname{curl} F \cdot n \, ds = \int_\gamma F \cdot dx,$$

$\boxed{\text{we see that the flux of } \operatorname{curl} F \text{ is equal, for } A, B}$

c) We must compute $\cancel{\oiint} \int_\gamma F \cdot dx$

for $\gamma(t) = (\cos t, \sin t, 0)$ $0 \le t \le 2\pi$

$= \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0) \, dt = \boxed{0}$