Math 21a Supplement on Charge Density

Surface integrals arise in the physics of the manner in which electric charge is distributed over the surface of an electrically charged object. To a first approximation, the story goes as follows: A typical solid object (think of a ball, pen, comb, ...) that you pick up will be electrically neutral. This is to say that it contains the same number of negative charges as positive ones. Here, the negative charges are carried by electrons from the constituent atoms, while the positive charges are carried by the protons which make up (in part) the nuclei of the constituent atoms. Moreover, since like charges exert a mutually repelling force, there will be local neutrality also. This is to say that the net charge in any macroscopic region of the object will also be zero.

Now, it turns out that there are various ways to add electrons to objects or take away electrons and thus give the object a net electric charge. For example, rubbing a plastic comb on a dry day will give the comb a net charge. (You know the resulting electric field as 'static electricity'.) In any event, if an object is given a net charge, the question arises as to how that charge is distributed through the object. In this regard, the key point is that like charges repel, so each charge will try to get as far as possible from the others. The result of this is that the excess charge will be, to a very good approximation, localized on the surface of the object. Thus, the result is a distribution of charges on the surface which can be described (again, to a good approximation) by a function which assigns to each point on the surface the charge density at that point. If this function is called σ, then the total charge on the body is computed by the surface integral

$$\iint_S \sigma \quad , \tag{1}$$

where S signifies the surface of the object. Likewise, the total charge on some portion, S_0, of the surface S can be computed by integrating the function σ over only S_0.

With regard to the form of the charge density function σ, one might guess at first that σ should be the constant function. Indeed, this is typically correct for balls. But if the surface is not completely symmetric, then there is no reason for σ to be constant. Indeed, if the surface has ripples or bumps, then a constant charge density will put charges at the bends closer to more neighbors then charges on the flatter portions of the surface. Thus, one expects the charge density to be smaller near bends and bulges in the surface. The fact is that it is a very non-trivial task to calculate the expected charge density function for a non-symmetric surface.

In fact, even symmetric objects can, at times, have interesting charge density functions. Indeed, consider the case of ball of radius 1 and center at the origin in R^3. Now, put one negative charge (thus, one electron) at the point +2 units up the z-axis. Even thought the ball is neutral, the ball does contain electrons and protons and the electrons will be repelled by this negative charge, while the protons attracted. The result will be a redistribution of the charges on the ball with the result being a charge density function σ for the surface which is not zero. Even so, as no charges were added to the ball, the integral in (1) will be zero, but there will be a net positive charge to the z > 0 hemisphere and a net negative charge to the z < 0 hemisphere. That is,

$$\iint_{S_{z>0}} \sigma > 0 \, , \tag{2}$$

where $S_{z>0}$ denotes the z > 0 part of the sphere.

One can further argue on symmetry grounds that σ will only depend on the spherical angle ϕ (latitude) and not at all on the angle θ (longitude). However, the determination of the precise form of σ requires some sophisticated physics.