2. The simplest 1-variable differential equations

Suppose that you are interested in the values of some quantity, say p, as a function of time, t. A differential equation for p is an equation which equates the time derivative of p to some function of p.

a) The simplest equation

The simplest differential equation reads

$$\frac{d}{dt} p = 0 \tag{2.1}$$

which asserts that the quantity p is constant in time. A less trivial example would be

$$\frac{d}{dt} p = c \ , \tag{2.2}$$

where c is some constant, say 2 or -3.4 or 10^7. (When this equation arises in the real world, the constant c is usually determined by some experimental measurement.) This last equation asserts that the rate of change of p is constant with time. Equation (2.2) can be solved fairly easily by integrating both sides:

$$p(t) - p(0) = \int_0^t \frac{dp}{dt} = ct \ . \tag{2.3}$$

The first equality here is just the Fundamental Theorem of Calculus. Thus, the solution to (2.2) has

$$p(t) = p(0) + ct \ . \tag{2.4}$$

b) Equations which involve p and its derivative

The simplest differential equation which involves both p and its derivative has the general form

$$\frac{d}{dt} p = f(p) \tag{2.5}$$

where f is some given function of p. This equation asserts that the rate of change of p at a given time is determined by the value of p at that time. A very important example is

$$\frac{d}{dt} p = p\ , \tag{2.6}$$

and, more generally,

$$\frac{d}{dt} p = ap\ , \tag{2.7}$$

where a is some given number, say, a = 3 or -5.56 etc.

These last two equations assert that the rate of change of p is proportional to p itself, the constant a being the proportionality factor. Here is a sample context in which (2.7) arises: Suppose that p(t) represents the number of bacteria in a petri dish at time t and that the dish is well sugared, so that the bacteria don't lack for food. One expects that each bacteria will fission into two bacteria at a regular rate, say once every 20 minutes. If no bacteria die, then the rate of change of p is equal to .05p in units of bacteria per minute. Put differently, if there are p(t) bacteria at time t, then one expects that roughly one out of every 20 bacteria are undergoing fission at any given minute. Thus, the population of bacteria in the petri dish at time t is governed by Equation (2.7) with a = .05 bacteria per minute.

c) The solution in the general case

There is always a solution to (2.5). Indeed, here is a general fact:

Choose any starting value for p(0) *and there exists a unique solution,*
p(t), to (2.5) which has the given starting value at t = 0.

(2.8)

Unfortunately, for a complicated function f in (2.5), there will not be a closed form expression for the solution p(t). Even so, one can in principle find all solutions to (2.5) for a given choice of f. Here is how: First, introduce F(p) to denote an anti-derivative of the function 1/f(p). Then the solution p(t) to (2.5) obeys the following non-differential equation at all times t:

$$F(p(t)) = t + c \,. \tag{2.9}$$

Here, c is a constant which is determined by the chosen starting value for p(t). Indeed, if you substitute t = 0 in (2.9), you find that c = F(p(0)).

By the way, you can verify that (2.9) must hold by differentiating both sides with respect to t while using the Chain rule to write $F(p(t))' = p'(t)/f(p(t))$.

Unfortunately, (2.9) says that t = F(p) which gives t as a function of p, rather than the desired p as a function of t. In principle, you can get from t as a function of p to p as a function of t, but if F is complicated, there won't be a closed form expression for the result.

As an aside, note that the derivation of p(t) from t(p) can be done graphically: First graph t as a function of p on paper with the vertical axis labeled t and the horizontal axis labeled p. Then, to find the value of p at a given time t, take the p value of the point where the horizontal line through (0, t) intersects the graph. The following figure gives an example of this technique:

$$(2.10)$$

Note that you need to be careful with your interpretation of the result when the horizontal line through (0, t) intersects the graph in more than one point!

d) The case where f(p) = ap

Probably the most important case of (2.5) is the case presented in (2.7) where f(p) = ap with a = constant. According to (2.9), the general solution to (2.7) is

$$p(t) = p(0)e^{at} \text{ .} \tag{2.11}$$

Here, p(0) is the value of p at time 0. Alternately, you can write (2.11) as

$$p(t) = p(t_0)e^{a(t - t_0)} \text{ ,} \tag{2.12}$$

where t_0 is any convenient time and $p(t_0)$ is the value of p at that time. Do you believe that (2.11) and (2.12) are the same? Don't let me con you. Check that they are the same by using (2.11) to solve for $p(t_0)$ and then plugging the result into (2.12).

By the way, notice that when a > 0, the quantity p(t) is increasing with time, and when a < 0, then p(t) decreases with time. These are important features of (2.7) which play a key role in their applications.

The validity of (2.11) (or (2.12)) can be established by plugging the right hand side of (2.11) (or (2.12)) into (2.7) and differentiating. Here, you should remember that

$$\frac{d}{dt} e^{at} = a\, e^{at} . \tag{2.13}$$

Because (2.11) is an exponential of t, the equation in (2.7) is often called the exponential growth equation.

e) When does the exponential growth equation arise?

The exponential growth equation $\frac{d}{dt} p = ap$ occurs ubiquitously in the sciences. There are two reasons for this. Here is the first reason: Suppose that p(t) represents the population of identical particles or creatures at time t which do not interact with each other. Use a_b to denote the birth rate and a_d to denote the death rate. Then, the population p(t) will obey an exponential equation where the coefficient a is equal to the difference, $a_b - a_d$, (Note that if a_b is the birth rate (measured, say in births per second), then $\frac{1}{a_b}$ is the average time between births. Likewise, $\frac{1}{a_d}$ is the average time between deaths. For example, if the birth rate is 4 per/day, then, all things being equal, on average there will be a birth every quarter day.)

Notice that these quantities a_b and a_d, and hence the quantity a can, in principle, be determined by experiments on some small number of creatures (or particles) in isolation. This is what makes the exponential equation so useful. Experiments done with small numbers of creatures or particles can be used to predict the behavior of large numbers of particles. This is the great utility of (2.7) and (2.11). They allow you to predict the behavior of large numbers of creatures or particles in terms of quantities which have been measured from experiments with small numbers of particles. However, the large numbers of particles or creatures must not interact with each other in a substantive way. If they do, then all bets are off vis a` vis the applicability of (2.7) or (2.11).

The second reason for the ubiquitous appearance of (2.7) comes via a strategy which involves the replacement of a complicated function by its linear approximation near a point of interest.

To elaborate on this strategy, remark that functions can be arbitrarily complicated. For example, consider the function

$$f(x) = \cos(\sin(\frac{3}{\sin(x^3) + 4 + x\cos x})) \ . \tag{2.14}$$

What would you do with the differential equation

$$\frac{d}{dt} x = f(x) \tag{2.15}$$

for a function x of t, where f is given by (2.14)?

(Note that here the unknown function is x and not p nor even q. Remember that there is no universal ‘name’ for the unknown function and so I (and you) can name it what you like.)

The replacement of f by its linear approximation allows you to deal with a horrible function f in (2.15). However, be aware that there is a cost to making this replacement; I discuss the cost below.

The linear approximation replacement can be useful in the case that you are only interested in the solution x(t) to (2.15) only for x near some point x_0 of interest. If this happens to be the case (and often it is), then you only need to know about the function f near the point x_0. For example, x_0 might be zero, or it might be 12.33 or -21.677.

In any event, the strategy is to sacrifice some accuracy for solvability. You replace the horrible function f(x) with an approximate function, a function L(x) which is very close to f(x) as long as x is close to x_0. The function L(x) should be such that you can actually solve the equation

$$\frac{d}{dt} x = L(x) \quad . \tag{2.16}$$

(Otherwise, why go through the trouble?)

Here is the key point:

The solution, x(t), *to (2.16) will behave much like the solution to (2.15) for those times t where x(t) from (2.16) is near to* x_0.

That is, you are interested in the solution to (2.15), but this equation is too hard to solve. So, instead, you solve an easier equation, (2.16), and observe that for times t where the solution to (2.16) is near to x_0, then the solution to (2.16) will provide a reasonable approximation to the desired solution to (2.15). Thus, you will gain some knowledge at the expense of some accuracy. The cost in making the replacement of (2.15) by the approximation, (2.16), is accuracy. The gain is (some) knowledge. Whether the gain is worth the cost depends on the circumstances.

With the preceding understood, consider using for L(x) in (2.16) the linear approximation to the given function f(x) in the vicinity of the point x_0 of interest. Thus,

$$L(x) = f(x_0) + f'(x_0)\,(x - x_0) \; . \tag{2.17}$$

In this case, (2.16) reads

$$\frac{d}{dt} x = f(x_0) + f'(x_0)\,(x - x_0)\ .$$

(2.18)

In the case where $f'(x_0) = 0$, this last equation is the same as equation (1) with $c = f(x_0)$, in which case the solution is

$$x(t) = x_0 + f(x_0)\, t\ .$$

(2.19)

In the case where $f'(x_0) \neq 0$, the equation in (2.18) is, after a change of variables, just the exponential equation from (2.7) with $a = f'(x_0)$. Indeed, introduce

$$u(t) = x(t) - x_0 + f(x_0)/f'(x_0).$$

(2.20)

Then, the right hand side of (2.18) is just $f'(x_0)\, u$. Meanwhile, as x_0 and $f(x_0)/f'(x_0)$ are constants, the time derivative of $u(t)$ is the same as that of $x(t)$. And, with this understood, one sees (2.18) as being equivalent to

$$\frac{d}{dt} u = f'(x_0)\, u\ .$$

(2.21)

This is just the exponential equation, with solution $u = u(0)\, e^{f'(x_0)\, t}$. Now, use (2.20) to write u in terms of x and so discover that

$$x(t) = x_0 + (e^{f'(x_0)\, t} - 1)\, f(x_0)/f'(x_0)\ .$$

(2.22)

I want to stress that this is not a solution to the original equation (2.16). However, as long as t is small (so that x(t) is nearly x_0), the function x(t) in (2.22) behaves very much like the solution to (2.16) which starts at x_0 when t = 0.