Math 21a Handout on Lagrange Multipliers

The purpose of this handout is to supply some additional examples of the Lagrange multiplier method for solving constrained equations for three unknowns. In this regard, remember that the basic problem is to find the maxima, minima and/or stationary points of some given function f(x, y, z) on R^3 subject to the constraint that g(x, y, z) = 0 where g is another function.

Note that these constrained problems typically arise in one of two ways: First, they can arise when looking for the extreme points of a function in some region of space; for in this case a search must be made for extreme points both inside the region and on its boundary. In particular, the problem of finding the extreme points on the boundary can be phrased as a constrained extremal problem. For example, imagine that the region in question is the solid Earth, and the function in question gives the temperature at each point. Then, a search for extreme points must look for interior extreme points and also extreme points on the surface.

Constrained extremal problems also arise when looking for the stationary points of a given function's restriction to some given surface in R^3. For example, take the surface in question to be the surface of the earth, and consider finding the extreme points of the function whose values measure a year's average of sun light intensity at each point.

Anyway, what follows are some sample constrained problems.

<u>Example</u> <u>1</u>: Model the surface of the Earth as the sphere where $x^2 + y^2 + z^2 = 1$. Then, suppose that the average temperature at each point on the surface is given by the values of the function $T = x^2 + 2xy + y^2 - 2z^2$. Find the points with the maximum and also the minimum temperature,

Here is the solution: The constraint function is $g(x, y, z) \equiv x^2 + y^2 + z^2 - 1$, and the extreme points occur where $\nabla T = \lambda \nabla g$ for some constant λ. A computation finds:

- $\nabla g = (2x, 2y, 2z)$.
- $\nabla T = (2x + 2y, 2y + 2x, - 4z)$.

(1)

Thus, the equation $\nabla T = \lambda \nabla g$ holds if and only if

- $2x + 2y = \lambda\, 2x$.
- $2y + 2x = \lambda\, 2y$.
- $-4z = \lambda\, 2z$.

(2)

The last point in (2) can hold only if $\lambda = -2$ or $z = 0$. I'll treat these two possibilities in turn.

Consider the case $\lambda = -2$. Then, the first and second points in (2) read

- $2x + 2y = -4x$.
- $2y + 2x = -4y$.

(3)

The latter points can be rephrased to read

- $6x = -2y$.
- $6y = -2x$.

(4)

And, this first line says that $x = - y/3$, while the second says that $x = -3y$. Clearly, these two assertions are compatible only if $x = y = 0$. Finally, this last forces $z = \pm 1$ since $x^2 + y^2 + z^2 = 1$. Thus, I have found two stationary points, (0, 0, 1) and (0, 0, -1).

Now consider the case where $z = 0$. In this case, the first two points in (2) can be rephrased to read

- $(1 - \lambda) x = -y$.
- $(1 - \lambda) y = -x$.

(5)

Now, it is not possible for λ to equal 1 since then (5) says that $x = y = 0$ and with $z = 0$, this would violate the constraint condition $x^2 + y^2 + z^2 = 1$. Then, as $\lambda \neq 1$, the first point in (5) implies that $x = -y /(1 - \lambda)$. Meanwhile, the second point gives $x = -(1 - \lambda)y$. Now, you might think that these two conditions on x can be solved by setting $y = 0$; but then x would have to vanish too and with $z = 0$, the constraint $x^2 + y^2 + z^2 = 1$ would be violated. So with $y \neq 0$, the only way to solve these constraints is if $1/(1 - \lambda) = (1 - \lambda)$. This last equation holds if and only if $\lambda = 0$ or $\lambda = 2$. In the case $\lambda = 0$, both points of (5) read $x = -y$. Since $z = 0$, this last condition plus the constraint $x^2 + y^2 = 1$ force $(x, y) = (1/\sqrt{2}, -1/\sqrt{2})$ or $(-1/\sqrt{2}, 1/\sqrt{2})$. On the other hand, in the case $\lambda = 2$, both points of (5) read $x = y$ and, since $z = 0$, the constraint $x^2 + y^2 = 1$ implies that $(x, y) = (1/\sqrt{2}, 1/\sqrt{2})$ or $(-1/\sqrt{2}, -1/\sqrt{2})$.

To summarize: There are six stationary points for this extremal problem:

$$(0, 0, 1), (0, 0, -1), (1/\sqrt{2}, -1/\sqrt{2}, 0), (-1/\sqrt{2}, 1/\sqrt{2}, 0), (1/\sqrt{2}, 1/\sqrt{2}, 0), (-1/\sqrt{2}, -1/\sqrt{2}, 0) \qquad (6)$$

Substituting these values into the function T shows that the first two are minima and the last two are maxima. The middle pair are stationary points that are neither local minima nor local maxima.

Example 2: Consider the problem of finding local maxima and minima of $f = x^2 - y^2 + z^2/(1 + z^2)$ on the infinite cylinder where $x^2 + y^2 = 1$ and $-\infty < z < \infty$. In this example, the constraint function is $g(x, y, z) = x^2 + y^2 - 1$ and the components of the Lagrange multiplier equation $\nabla f = \lambda \nabla g$ read

- $2x = \lambda\, 2x$.
- $-2y = \lambda\, 2y$.
- $2z/(1 + z^2)^2 = 0$.

(7)

Here is how to analyze (7): First off, the last point in (7) requires $z = 0$. Next, observe that the first point in (7) requires either $x = 0$ or $\lambda = 1$. If $x = 0$, then the constraint $x^2 + y^2 = 1$ forces $y = \pm 1$, and thus (0, 1, 0) and (0, -1, 0) are stationary points. On the other hand, if $x \neq 0$, then $\lambda = 1$ and the second point requires $y = 0$. In this case, the constraint $x^2 + y^2 = 1$ force $x = \pm 1$. Thus, (1, 0 , 0) and (-1, 0, 0) are also stationary points. Hence, there are four stationary points in all. Substituting these into the expression for f finds that (0, ±1, 0) are local minima while (±1, 0, 0) are saddles.

Example 3: In this case, the consider the problem of finding all stationary points of the function $f(x, y, z) = -x^2 + z^2$ on the paraboloid given by the constraint $g(x, y, z) = z - x^2 - y^2 = 0$. In this case, the three components of the Lagrange multiplier equation $\nabla f = \lambda \nabla g$ reads

- $-2x = -\lambda\, 2x$.
- $0 = -\lambda\, 2y$.
- $2z = \lambda$.

(8)

To find the solutions to (8), note that the middle point requires either $\lambda = 0$ or $y = 0$. In the former case, the third point requires $z = 0$ and the first requires $x = 0$. Then, the constraint equation $z = x^2 + y^2$ forces $y = 0$. So, (0, 0, 0) is a stationary point.

Now consider the case $y = 0$ solving the middle point in (8). The first point in (8) requires either $x = 0$ or $\lambda = 1$. The former possibility gives the point (0, 0, 0) as a stationary point (again) since $x = y = 0$ and the constraint equation $z = x^2 + y^2$ force z to vanish too. On the other hand, if $x \neq 0$ and $\lambda = 1$, then the third point in (8) requires $z = 1/2$. Since $y = 0$, the constraint equation reads $z = x^2$ so $x = \pm 1/\sqrt{2}$. Thus, $(1/\sqrt{2}, 0, 1/2)$ and $(-1/\sqrt{2}, 0 , 1/2)$ are also stationary points.

Plugging into $f = -x^2 + z^2$ the values of x and z from these three stationary points finds that $(\pm 1/\sqrt{2}, 0, 1/2)$ are global minima, while (0, 0, 0) is neither a local minimum or maximum.