

# Math 21a PDE Handout

## Suggested Problems - Answers Spring 2002

- (1) (a)  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (6x) + (-6x) = 0$  so it is a solution
- (b)  $(12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0$  yes, it's a solution
- (c)  $(-2 \sin(x-y) e^{k-y}) + (-2 \sin(x-y) e^{k-y}) \neq 0$  not a solution
- (d)  $\left(\frac{2xy}{(x^2+y^2)^2}\right) + \left(\frac{-2xy}{(x^2+y^2)^2}\right) = 0$  yes it's a solution
- (e)  $(3998y) + (4004x) \neq 0$  so no, not a solution
- (f)  $(e^x (\sin y + 2 \cos y)) + (e^x (-\sin y - 2 \cos y)) = 0$ , it is a solution
- (2) (a)  $\frac{\partial^2 u}{\partial t^2} = -4 (\sin(x+2t) + \cos(x-2t)) = 4 \frac{\partial^2 u}{\partial x^2}$ , yes it's a solution with  $c^2 = 4$   
(then assuming  $c$  positive,  $c = 2$ )
- (b)  $\frac{\partial^2 u}{\partial t^2} = -\frac{1}{t^2} \neq c \frac{\partial^2 u}{\partial x^2}$  for any constant  $c$ , so it's not a solution
- (c)  $\frac{\partial^2 u}{\partial t^2} = 24x = \frac{\partial^2 u}{\partial x^2}$  so, yes, it's a solution with  $c^2 = 1$   
...  $c = 1$
- (d)  $\frac{\partial^2 u}{\partial t^2} = -100^2 \sin(100x) \sin(100t) = 100^2 \frac{\partial^2 u}{\partial x^2}$ , is a solution,  $c = 100$
- (e)  $\frac{\partial^2 u}{\partial t^2} = 4004x \neq c \frac{\partial^2 u}{\partial x^2}$ , so not a solution
- (f)  $\frac{\partial^2 u}{\partial t^2} = 8 = 4 \frac{\partial^2 u}{\partial x^2}$ , so yes, it's a solution with  $c^2 = 4$   
( $c = 2$ )
- (3) (a) here the only derivative is with respect to  $x$ , so we can treat this as a simple ordinary differential equation, and treat  $y$  as just an extra parameter - ending up multiplying by an arbitrary function  $g(y)$ ...
- so if we saw  $\frac{d^2 f}{dx^2} + 9f = 0$  (where  $f$  is a fct. just of  $x$ )
- Then the solution would be  $A \cos(3x) + B \sin(3x)$   
(from Math 1b section on differential equations  $\rightarrow$  this is the general solution to an undamped spring equation)

(3) (a) continued -- then the general solution for  $\frac{\partial^2 u}{\partial x^2} + qu = 0$  is just  $(A \cos(\sqrt{q}x) + B \sin(\sqrt{q}x)) g(y)$  where  $A, B$  are arbitrary constants, and  $g(y)$  is a function of  $y$  alone

(b) This problem is extremely difficult as stated! My apologies for not having stated it correctly as just  $\frac{\partial u}{\partial y} + 2yu = 0$  (not  $\frac{\partial^2 u}{\partial y^2}$ !) In the correct form one solves this as a simple ODE in the variable  $y$ : solution  $Ce^{-y^2}$  then multiply this by an arbitrary function  $g(x)$ , of  $x$  alone, so general solution is  $g(x)e^{-y^2}$

(c)  $\frac{\partial u}{\partial x} = 2xyu$ , again solve as an ODE in  $x$  alone, then the solution is determined up to a multiple by a function in  $y$ . Again, basic Math 1b technique solves the ODE: divide by  $u$ , "multiply" by  $dx$ ...

$$\text{get } \int \frac{1}{u} du = \int 2xy dx, \text{ so } \ln|u| = x^2 y + C_1$$

$$\text{then as an ODE: } u = \pm e^{x^2 y + C_1} = C e^{x^2 y}$$

so with  $u$  a function of  $x$  and  $y$   $u(x, y) = g(y) e^{x^2 y}$

(d)  $\frac{\partial^2 u}{\partial x \partial y} = 0$  just integrate with respect to  $x$ :  
get  $\frac{\partial u}{\partial y} = f(y)$  (since then  $\frac{\partial}{\partial x}(f(y)) = 0$ )  
so  $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = 0$

then integrate with respect

to  $y$ : get  $\int f(y) dy = g(y) + h(x)$  (where  $\frac{d}{dy}(g(y)) = f(y)$ )

so general solution is just  $u(x, y) = g(y) + h(x)$ , where  $g$  and  $h$  are arbitrary functions of  $y$  and  $x$  alone, respectively

(4) (a) since  $\frac{\partial^2 u}{\partial x^2} = 0$ , then from the perspective of the variable  $x$ ,  $u(x, y) = ax + b$ ,

where  $a$  and  $b$  could be functions of  $y$ , i.e.,  $u(x, y) = a(y)x + b(y)$ . Next, since  $\frac{\partial^2 u}{\partial y^2} = 0$ , then  $a(y)$  and  $b(y)$  can at most be linear functions in  $y$ , i.e.  $a(y) = cy + d$ ,  $b(y) = fy + g$ , so...

(4)(a) the general solution is  $u(x,y) = (cy+d)x + fy + g$   
 $= cxy + dx + fy + g$ ,

where  $c, d, f, g$  are arbitrary constants.

(b) more straightforward than (4)(a):

since  $\frac{\partial u}{\partial x} = 0$ , then  $u(x,y) = f(y)$ , where  $f$  is a function of  $y$  alone. but  $\frac{\partial u}{\partial y} = 0$  so  $f(y) = k$ , a constant, so  $u(x,y) = k$  some constant

(c) since  $\frac{\partial^2 u}{\partial x^2} = 0$ , then  $u(x,y) = a(y)x + b(y)$

again, as in (4)(a) then  $\frac{\partial^2}{\partial x \partial y} (a(y)x + b(y)) = a'(y)$

so  $a'(y) = 0$  which implies  $a(y) = k$ , in which case  $u(x,y) = kx + b(y)$ ,  $k$  a constant,  $b$  a function of  $y$

(d) since  $\frac{\partial^2 u}{\partial x^2} = 0 = \frac{\partial^2 u}{\partial y^2}$ , then we already know

from (4)(a) that the general solution to this set of PDE's is  $cxy + dx + fy + g$ . Now add in the extra condition  $\frac{\partial^2 u}{\partial y \partial x} = 0$ :  $\frac{\partial^2}{\partial y \partial x} (cxy + dx + fy + g) = c$

so  $c = 0$ , and  $u(x,y) = dx + fy + g$ ,  $d, f, g$  constants

(5) We start by knowing the general solution to the wave equation is  $\left( A \cos\left(\frac{cn\pi}{l}t\right) + B \sin\left(\frac{cn\pi}{l}t\right) \right) \sin\left(\frac{n\pi x}{l}\right)$

(Note - on the final you will be given this solution if you need to use it - you don't need to memorize this!!)

Now length  $l = \pi$ , and  $c = 1$  (we'll assume  $c$  is positive) so in these four questions we'll be working with the general solution

$$u(x,t) = \left( A \cos(nt) + B \sin(nt) \right) \sin(nx)$$

where  $A, B$  are constants,  $n$  an integer

(a) We need  $u(x,0) = 0$  and  $u_t(x,0) = \sin(4x)$ , so start with  $u(x,0) = (A \cos(n \cdot 0) + B \sin(n \cdot 0)) \sin(nx) = (A + B \cdot 0) \sin(nx) = A \sin(nx)$

since  $u(x,0) = 0$ , then  $A = 0$

(5) (a) continued so  $u(x,t) = B \sin(nt) \sin(nx)$ .  
 Next  $u_t(x,0) = nB \cos(n \cdot 0) \sin(nx) = nB \sin(nx)$   
 so  $u_t(x,0) = \sin(4x)$  means  $n=4$ ,  $B = \frac{1}{4}$ ,  
 so  $u(x,t) = \frac{1}{4} \sin(4t) \sin(4x)$

(b) here everything is the same as in (a) up to  
 $u_t(x,0) = nB \sin(nx)$  now equals  $-0.1 \sin(2x)$   
 so  $n=2$ ,  $B = -0.05$ ,  $u(x,t) = -0.05 \sin(2t) \sin(2x)$

(c) now  $u(x,0) = A \sin(nx)$  is supposed to equal  $0.1 \sin(x)$   
 so  $A=0.1$ ,  $n=1$ , and  $u(x,t) = (0.1 \cos(t) + B \sin(t)) \sin(x)$   
 next  $u_t(x,0) = (-0.1 \sin(0) + B \cos(0)) \sin(x) = B \sin(x)$   
 so since it's supposed to equal  $-0.2 \sin(x)$ ,  
 then  $B = -0.2$ , and  $u(x,t) = (0.1 \cos(t) - 0.2 \sin(t)) \sin(x)$

(d) now  $u(x,0) = A \sin(nx) = 0.3 \sin(6x)$  so  $A=0.3$ ,  $n=6$   
 next  $u_t(x,0) = (-0.3 \sin(0) + B \cos(0)) \sin(6x) = B \sin(6x)$   
 so  $B = -0.2$ , and  $u(x,t) = (0.3 \cos(6t) - 0.2 \sin(6t)) \sin(6x)$

(6) To use the separation of variables technique we  
 assume that  $u(x,y) = F(x)G(y)$ , so

$$\frac{\partial u}{\partial x} = F'(x)G(y) \quad \text{and} \quad \frac{\partial u}{\partial y} = F(x)G'(y)$$

(a) so  $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = F'(x)G(y) - F(x)G'(y) = 0$ ,  
 so  $F'(x)G(y) = F(x)G'(y)$ , or  $\frac{F'(x)}{F(x)} = \frac{G'(y)}{G(y)}$

we know that this must equal a constant,  $k$ ,  
 (check the text if you are confused by this!)

so  $F'(x) = kF(x)$  and  $G'(y) = kG(y)$   
 then  $F(x) = C_1 e^{kx}$ ,  $G(y) = C_2 e^{ky}$ , (both by Math 1b, ODEs)

so  $u(x,y) = F(x)G(y) = C e^{kx+ky} = C e^{k(x+y)}$

$C, k$  arbitrary constants

(b) so  $F'(x)G(y) - y F(x)G'(y)$ , so  $\frac{F'(x)}{F(x)} = \frac{y G'(y)}{G(y)} = k$

(6) (b) continued, so  $F'(x) = kF(x)$ ,  $yG'(y) = kG(y)$

so  $F(x) = C_1 e^{kx}$ , and  $G(y) = C_2 y^k$

(Math 1b)

(trial and error - if this were on the final, you would probably be given this solution - don't worry!)

so  $u(x,y) = C(e^{kx})y^k$

(c) so  $x F'(x)G(y) - y F(x)G'(y) = 0$ , so  $\frac{x F'(x)}{F(x)} = \frac{y G'(y)}{G(y)} = k$

same as in (b), then  $F(x) = C_1 x^k$ ,  $G(y) = C_2 y^k$

so  $u(x,y) = C(xy)^k$

(d) now  $y F'(x)G(y) - x F(x)G'(y) = 0$ , so  $\frac{F'(x)}{x F(x)} = \frac{G'(y)}{y G(y)} = k$

so  $F'(x) = kx F(x)$ ,  $G'(y) = ky G(y)$

solutions:  $\int \frac{1}{F} dF = \int kx dx \dots \ln|F| = \frac{k}{2}x^2 + c$ ,  $F(x) = C_1 e^{\frac{k}{2}x^2}$

similarly  $G(y) = C_2 e^{\frac{k}{2}y^2}$ , so  $u(x,y) = C e^{\frac{k}{2}(x^2+y^2)}$

(e) here  $F'(x)G(y) + F(x)G'(y) = 2(x+y)F(x)G(y) = 2x F(x)G(y) + 2y F(x)G(y)$

then splitting into  $x, y$ :  $(F'(x) - 2x F(x))G(y) + F(x)(G'(y) - 2y G(y)) = 0$

so  $\frac{F'(x) - 2x F(x)}{F(x)} = \frac{G'(y) - 2y G(y)}{G(y)} = k$ , and  $F'(x) - 2x F(x) = k F(x)$   
 $G'(y) - 2y G(y) = -k G(y)$

so  $F'(x) = F(x)(2x+k)$ ,  $G'(y) = G(y)(2y-k)$

solutions:  $\int \frac{1}{F} dF = \int (2x+k) dx$  leads to  $F(x) = C e^{x^2+kx}$

so  $u(x,y) = C e^{x^2+kx} e^{y^2-ky} = C e^{x^2+y^2+k(x-y)}$

(f) so  $\frac{\partial^2 u}{\partial x \partial y} = F'(x)G'(y) = u = F(x)G(y)$ , so  $\frac{F'(x)}{F(x)} = \frac{G'(y)}{G(y)} = k$

then  $F'(x) = k F(x) \Rightarrow F(x) = C_1 e^{kx}$

and  $G'(y) = \frac{1}{k} G(y) \Rightarrow G(y) = C_2 e^{\frac{1}{k}y}$

so  $u(x,y) = C e^{(kx + \frac{1}{k}y)}$