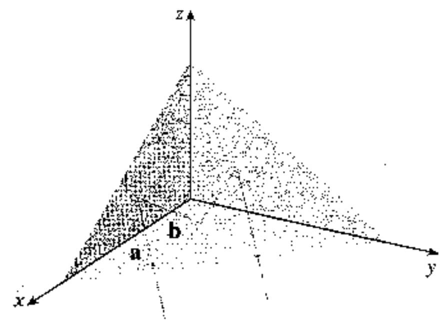
[illegible]. Suppose the three coordinate planes are all mirrored and a light ray given by the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ first strikes the xz-plane, as shown in the figure. Use the fact that the angle of incidence equals the angle of reflection to show that the direction of the reflected ray is given by $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. Deduce that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray. (American space scientists used this principle, together with laser beams and an array of corner mirrors on the Moon, to calculate very precisely the distance from Earth to the Moon.)



9.3 The Dot Product

So far we have added two vectors and multiplied a vector by a scalar. The question arises: Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, which we consider in this section. Another is the cross product, which is discussed in the next section.

Work and the Dot Product

An example of a situation in physics and engineering where we need to combine two vectors occurs in calculating the work done by a force. In Section 6.5 we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 1. If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. So here we have two vectors: the force $\mathbf{F}$ and the displacement $\mathbf{D}$. The **work** done by $\mathbf{F}$ is defined as the magnitude of the displacement, $|\mathbf{D}|$, multiplied by the magnitude of the applied force in the direction of the motion, which, from Figure 1, is

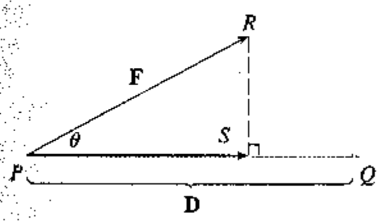


FIGURE 1

$$|\overrightarrow{PS}| = |\mathbf{F}| \cos\theta$$

So the work done by $\mathbf{F}$ is defined to be

$$W = |\mathbf{D}|\,(|\mathbf{F}|\cos\theta) = |\mathbf{F}|\,|\mathbf{D}|\cos\theta \qquad \boxed{1}$$

Notice that work is a scalar quantity; it has no direction. But its value depends on the angle between the force and displacement vectors.

We use the expression in Equation 1 to define the dot product of two vectors even when they don't represent force or displacement.

Definition The **dot product** of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leqslant \theta \leqslant \pi$. (So θ is the smaller angle between the vectors when they are drawn with the same initial point.) If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, we define $\mathbf{a} \cdot \mathbf{b} = 0$.

This product is called the **dot product** because of the dot in the notation $\mathbf{a} \cdot \mathbf{b}$. The result of computing $\mathbf{a} \cdot \mathbf{b}$ is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product**.

In the example of finding the work done by a force $\mathbf{F}$ in moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ by calculating $\mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos\theta$, it makes no sense for the angle θ between $\mathbf{F}$ and $\mathbf{D}$ to be $\pi/2$ or larger because movement from P to Q couldn't take place. We make no such restriction in our general definition of $\mathbf{a} \cdot \mathbf{b}$, however, and allow θ to be any angle from 0 to π.

EXAMPLE 1 If the vectors $\mathbf{a}$ and $\mathbf{b}$ have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

SOLUTION According to the definition,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/3) = 4 \cdot 6 \cdot \tfrac{1}{2} = 12$$ ■

EXAMPLE 2 A crate is hauled 8 m up a ramp under a constant force of 200 N applied at an angle of 25° to the ramp. Find the work done.

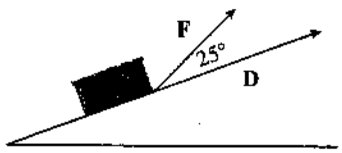


FIGURE 2

SOLUTION If $\mathbf{F}$ and $\mathbf{D}$ are the force and displacement vectors, as pictured in Figure 2, then the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos 25°$$

$$= (200)(8) \cos 25° \approx 1450 \text{ N·m} = 1450 \text{ J}$$ ■

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. For such vectors we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$. The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors. Therefore

2 Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

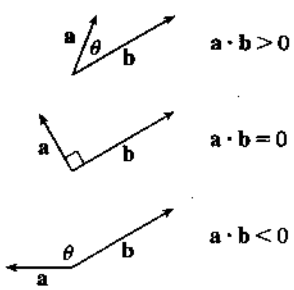


FIGURE 3

Because $\cos\theta > 0$ if $0 \leq \theta < \pi/2$ and $\cos\theta < 0$ if $\pi/2 < \theta \leq \pi$, we see that $\mathbf{a} \cdot \mathbf{b}$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of $\mathbf{a} \cdot \mathbf{b}$ as measuring the extent to which $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. The dot product $\mathbf{a} \cdot \mathbf{b}$ is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 3). In the extreme case where $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, we have $\theta = 0$, so $\cos\theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If $\mathbf{a}$ and $\mathbf{b}$ point in exactly opposite directions, then $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

The Dot Product in Component Form

Suppose we are given two vectors in component form:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \qquad \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

We want to find a convenient expression for $\mathbf{a} \cdot \mathbf{b}$ in terms of these components. If we apply the Law of Cosines to the triangle in Figure 4, we get

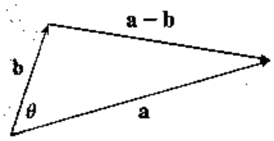


FIGURE 4

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} \end{aligned}$$

Solving for the dot product, we obtain

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \tfrac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2\right) \\ &= \tfrac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - (a_1 - b_1)^2 - (a_2 - b_2)^2 - (a_3 - b_3)^2\right] \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

> The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is
>
> $$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Thus, to find the dot product of **a** and **b** we multiply corresponding components and add. The dot product of two-dimensional vectors is found in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1b_1 + a_2b_2$$

EXAMPLE 3

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \left\langle 6, 2, -\tfrac{1}{2} \right\rangle = (-1)(6) + 7(2) + 4\left(-\tfrac{1}{2}\right) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$ ■

EXAMPLE 4 Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

SOLUTION Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by (2). ■

EXAMPLE 5 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

SOLUTION Let θ be the required angle. Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

we have, from the definition of the dot product

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84°)$$

EXAMPLE 6 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2\rangle$, so the work done is

$$\begin{aligned} W &= \mathbf{F}\cdot\mathbf{D} = \langle 3, 4, 5\rangle \cdot \langle 2, 5, 2\rangle \\ &= 6 + 20 + 10 = 36 \end{aligned}$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J.

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

Properties of the Dot Product If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$
5. $\mathbf{0}\cdot\mathbf{a} = 0$

Properties 1, 2, and 5 are immediate consequences of the definition of a dot product. Property 3 is best proved using components:

$$\begin{aligned} \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \langle a_1, a_2, a_3\rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3\rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} \end{aligned}$$

The proof of Property 4 is left as Exercise 39.

Projections

Figure 5 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then

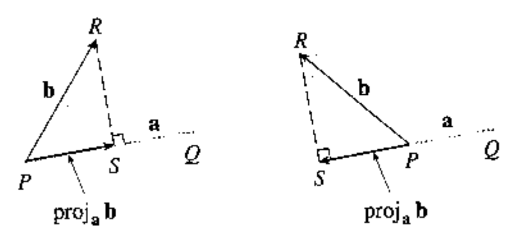


FIGURE 5
Vector projections

the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $\text{proj}_\mathbf{a}\,\mathbf{b}$. The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ (also called the **component** of **b along a**) is defined to be the magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. (See Figure 6; you can think of the scalar projection of $\mathbf{b}$ as being the length of a shadow of $\mathbf{b}$.) This is denoted by $\text{comp}_\mathbf{a}\,\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leqslant \pi$. (Note that we used the component of the force $\mathbf{F}$ along the displacement $\mathbf{D}$, $\text{comp}_\mathbf{D}\,\mathbf{F}$, at the beginning of this section.)

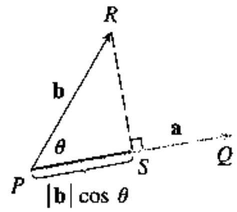


FIGURE 6
Scalar projection

The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of $\mathbf{a}$ and $\mathbf{b}$ can be interpreted as the length of $\mathbf{a}$ times the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of $\mathbf{b}$ along $\mathbf{a}$ can be computed by taking the dot product of $\mathbf{b}$ with the unit vector in the direction of $\mathbf{a}$. To summarize:

Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \text{comp}_\mathbf{a}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \text{proj}_\mathbf{a}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$

EXAMPLE 7 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{comp}_\mathbf{a}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of $\mathbf{a}$:

$$\text{proj}_\mathbf{a}\,\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle$$

At the beginning of this section we saw one use of projections in physics—we used a scalar projection of a force vector in defining work. Other uses of projections occur in three-dimensional geometry. In Exercise 33 you are asked to use a projection to find the distance from a point to a line, and in Section 9.5 we use a projection to find the distance from a point to a plane.

9.3 Exercises

1. Which of the following expressions are meaningful? Which are meaningless? Explain.
(a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ (b) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
(c) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ (d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
(e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$ (f) $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$

2. Find the dot product of two vectors if their lengths are 6 and $\frac{1}{3}$ and the angle between them is $\pi/4$.

3–8 ■ Find $\mathbf{a} \cdot \mathbf{b}$.

3. $|\mathbf{a}| = 12$, $|\mathbf{b}| = 15$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/6$

4. $\mathbf{a} = \langle \frac{1}{2}, 4 \rangle$, $\mathbf{b} = \langle -8, -3 \rangle$

5. $\mathbf{a} = \langle 5, 0, -2 \rangle$, $\mathbf{b} = \langle 3, -1, 10 \rangle$

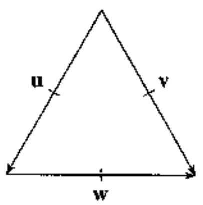
6. $\mathbf{a} = \langle s, 2s, 3s \rangle$, $\mathbf{b} = \langle t, -t, 5t \rangle$

7. $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 9\mathbf{k}$

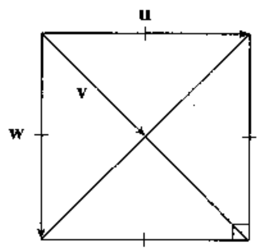
8. $\mathbf{a} = 4\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

9–10 ■ If $\mathbf{u}$ is a unit vector, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$.

9.



10.



11. (a) Show that $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.
(b) Show that $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.

12. A street vendor sells a hamburgers, b hot dogs, and c soft drinks on a given day. He charges \$2 for a hamburger, \$1.50 for a hot dog, and \$1 for a soft drink. If $\mathbf{A} = \langle a, b, c \rangle$ and $\mathbf{P} = \langle 2, 1.5, 1 \rangle$, what is the meaning of the dot product $\mathbf{A} \cdot \mathbf{P}$?

13–15 ■ Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

13. $\mathbf{a} = \langle 3, 4 \rangle$, $\mathbf{b} = \langle 5, 12 \rangle$

14. $\mathbf{a} = \langle 6, -3, 2 \rangle$, $\mathbf{b} = \langle 2, 1, -2 \rangle$

15. $\mathbf{a} = \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

16. Find, correct to the nearest degree, the three angles of the triangle with the vertices $P(0, -1, 6)$, $Q(2, 1, -3)$, and $R(5, 4, 2)$.

17. Determine whether the given vectors are orthogonal, parallel, or neither.
(a) $\mathbf{a} = \langle -5, 3, 7 \rangle$, $\mathbf{b} = \langle 6, -8, 2 \rangle$
(b) $\mathbf{a} = \langle 4, 6 \rangle$, $\mathbf{b} = \langle -3, 2 \rangle$
(c) $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
(d) $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

18. For what values of b are the vectors $\langle -6, b, 2 \rangle$ and $\langle b, b^2, b \rangle$ orthogonal?

19. Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

20. For what values of c is the angle between the vectors $\langle 1, 2, 1 \rangle$ and $\langle 1, 0, c \rangle$ equal to 60°?

21–24 ■ Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

21. $\mathbf{a} = \langle 2, 3 \rangle$, $\mathbf{b} = \langle 4, 1 \rangle$

22. $\mathbf{a} = \langle 3, -1 \rangle$, $\mathbf{b} = \langle 2, 3 \rangle$

23. $\mathbf{a} = \langle 4, 2, 0 \rangle$, $\mathbf{b} = \langle 1, 1, 1 \rangle$

24. $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

25. Show that the vector $\text{orth}_{\mathbf{a}}\,\mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\,\mathbf{b}$ is orthogonal to $\mathbf{a}$. (It is called an **orthogonal projection** of $\mathbf{b}$.)

26. For the vectors in Exercise 22, find $\text{orth}_{\mathbf{a}}\,\mathbf{b}$ and illustrate by drawing the vectors $\mathbf{a}$, $\mathbf{b}$, $\text{proj}_{\mathbf{a}}\,\mathbf{b}$, and $\text{orth}_{\mathbf{a}}\,\mathbf{b}$.

27. If $\mathbf{a} = \langle 3, 0, -1 \rangle$, find a vector $\mathbf{b}$ such that $\text{comp}_{\mathbf{a}}\,\mathbf{b} = 2$.

28. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors.
(a) Under what circumstances is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \text{comp}_{\mathbf{b}}\,\mathbf{a}$?
(b) Under what circumstances is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \text{proj}_{\mathbf{b}}\,\mathbf{a}$?

29. A constant force with vector representation $\mathbf{F} = 10\mathbf{i} + 18\mathbf{j} - 6\mathbf{k}$ moves an object along a straight line from the point (2, 3, 0) to the point (4, 9, 15). Find the work done if the distance is measured in meters and the magnitude of the force is measured in newtons.

30. Find the work done by a force of 20 lb acting in the direction N50°W in moving an object 4 ft due west.

31. A woman exerts a horizontal force of 25 lb on a crate as she pushes it up a ramp that is 10 ft long and inclined at an angle of 20° above the horizontal. Find the work done on the box.

32. A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 50 N. The handle of the wagon is held at an angle of 30° above the horizontal. How much work is done?

33. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

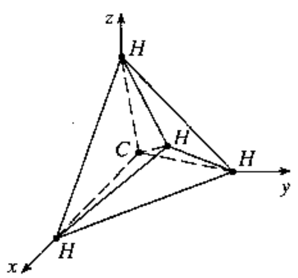
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

34. If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

35. Find the angle between a diagonal of a cube and one of its edges.

36. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

37. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5°. [*Hint:* Take the vertices of the tetrahedron to be the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$ as shown in the figure. Then the centroid is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.]



38. If $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are all nonzero vectors, show that $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

39. Prove Property 4 of the dot product. Use either the definition of a dot product (considering the cases $c > 0$, $c = 0$, and $c < 0$ separately) or the component form.

40. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

41. Prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|$$

42. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.
(b) Use the Cauchy-Schwarz Inequality from Exercise 41 to prove the Triangle Inequality. [*Hint:* Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

43. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.
(b) Prove the Parallelogram Law. (See the hint in Exercise 42.)

9.4 The Cross Product

The **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$, unlike the dot product, is a vector. For this reason it is also called the **vector product**. We will see that $\mathbf{a} \times \mathbf{b}$ is useful in geometry because it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. But we introduce this product by looking at a situation where it arises in physics and engineering.

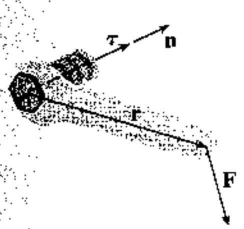


FIGURE 1

Torque and the Cross Product

If we tighten a bolt by applying a force to a wrench as in Figure 1, we produce a turning effect called a *torque* $\boldsymbol{\tau}$. The magnitude of the torque depends on two things:

- The distance from the axis of the bolt to the point where the force is applied. This is $|\mathbf{r}|$, the length of the position vector $\mathbf{r}$.
- The scalar component of the force $\mathbf{F}$ in the direction perpendicular to $\mathbf{r}$. This is the only component that can cause a rotation and, from Figure 2, we see that it is

$$|\mathbf{F}| \sin \theta$$

where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{F}$.

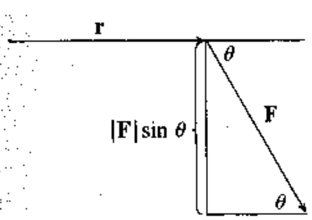


FIGURE 2

We define the magnitude of the torque vector to be the product of these two factors:

$$|\boldsymbol{\tau}| = |\mathbf{r}||\mathbf{F}| \sin \theta$$

The direction is along the axis of rotation. If $\mathbf{n}$ is a unit vector that points in the direction in which a right-threaded bolt moves (see Figure 1), we define the **torque** to be the vector

$$\boxed{1} \qquad \boldsymbol{\tau} = (|\mathbf{r}||\mathbf{F}| \sin \theta)\mathbf{n}$$

We denote this torque vector by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ and we call it the *cross product* or *vector product* of $\mathbf{r}$ and $\mathbf{F}$.

The type of expression in Equation 1 occurs so frequently in the study of fluid flow, planetary motion, and other areas of physics and engineering, that we define and study the cross product of *any* pair of three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$.

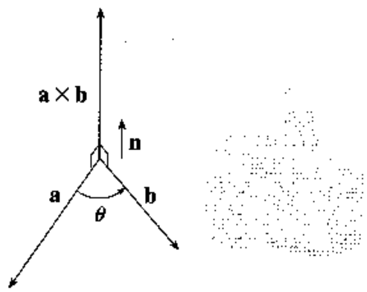


FIGURE 3
The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

Definition If $\mathbf{a}$ and $\mathbf{b}$ are nonzero three-dimensional vectors, the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin \theta)\mathbf{n}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leqslant \theta \leqslant \pi$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and whose direction is given by the **right-hand rule**: If the fingers of your right hand curl through the angle θ from $\mathbf{a}$ and $\mathbf{b}$, then your thumb points in the direction of $\mathbf{n}$. (See Figure 3.)

If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, then we define $\mathbf{a} \times \mathbf{b}$ to be $\mathbf{0}$.

Because $\mathbf{a} \times \mathbf{b}$ is a scalar multiple of $\mathbf{n}$, it has the same direction as $\mathbf{n}$ and so

$$\boxed{\mathbf{a} \times \mathbf{b} \text{ is orthogonal to both } \mathbf{a} \text{ and } \mathbf{b}}$$

Notice that two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if the angle between them is 0 or π. In either case, $\sin \theta = 0$ and so $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

▲ In particular, any vector $\mathbf{a}$ is parallel to itself, so

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

$$\boxed{\text{Two nonzero vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel if and only if } \mathbf{a} \times \mathbf{b} = \mathbf{0}.}$$

This makes sense in the torque interpretation: If we pull or push the wrench in the direction of its handle (so $\mathbf{F}$ is parallel to $\mathbf{r}$), we produce no torque.

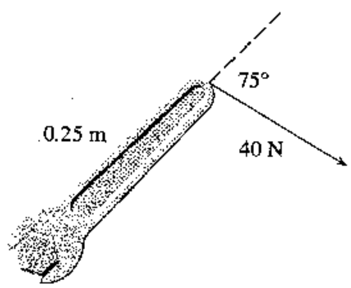


FIGURE 4

EXAMPLE 1 A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 4. Find the magnitude of the torque about the center of the bolt.

SOLUTION The magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin 75^\circ |\mathbf{n}| = (0.25)(40) \sin 75^\circ$$
$$= 10 \sin 75^\circ \approx 9.66 \text{ N·m} = 9.66 \text{ J}$$

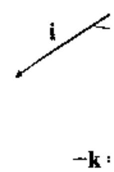


FIGURE 5

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\,\mathbf{n} \approx 9.66\mathbf{n}$$

where $\mathbf{n}$ is a unit vector directed down into the page.

EXAMPLE 2 Find $\mathbf{i} \times \mathbf{j}$ and $\mathbf{j} \times \mathbf{i}$.

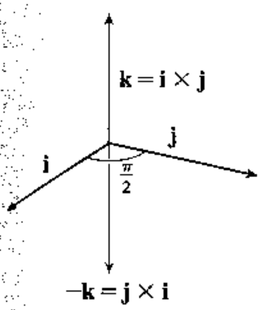


FIGURE 5

SOLUTION The standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ both have length 1 and the angle between them is $\pi/2$. By the right-hand rule, the unit vector perpendicular to $\mathbf{i}$ and $\mathbf{j}$ is $\mathbf{n} = \mathbf{k}$ (see Figure 5), so

$$\mathbf{i} \times \mathbf{j} = (|\mathbf{i}||\mathbf{j}|\sin(\pi/2))\mathbf{k} = \mathbf{k}$$

But if we apply the right-hand rule to the vectors $\mathbf{j}$ and $\mathbf{i}$ (in that order), we see that $\mathbf{n}$ points downward and so $\mathbf{n} = -\mathbf{k}$. Thus

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

From Example 2 we see that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

so the cross product is not commutative. Similar reasoning shows that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

In general, the right-hand rule shows that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Another algebraic law that fails for the cross product is the associative law for multiplication; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

For instance, if $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i}$, and $\mathbf{c} = \mathbf{j}$, then

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

whereas

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

However, some of the usual laws of algebra *do* hold for cross products:

Properties of the Cross Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Property 2 is proved by applying the definition of a cross product to each of the three expressions. Properties 3 and 4 (the Vector Distributive Laws) are more difficult to establish; we won't do so here.

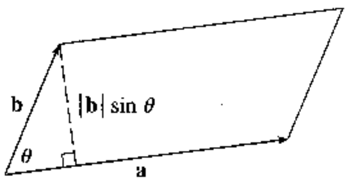


FIGURE 6

A geometric interpretation of the length of the cross product can be seen by looking at Figure 6. If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a}\times\mathbf{b}|$$

The length of the cross product $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.

The Cross Product in Component Form

Suppose **a** and **b** are given in component form:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

We can express $\mathbf{a}\times\mathbf{b}$ in component form by using the Vector Distributive Laws together with the results from Example 2:

▲ Note that

$\mathbf{i}\times\mathbf{i}=\mathbf{0} \quad \mathbf{j}\times\mathbf{j}=\mathbf{0} \quad \mathbf{k}\times\mathbf{k}=\mathbf{0}$

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\
&= a_1b_1\mathbf{i}\times\mathbf{i} + a_1b_2\mathbf{i}\times\mathbf{j} + a_1b_3\mathbf{i}\times\mathbf{k}\\
&\qquad + a_2b_1\mathbf{j}\times\mathbf{i} + a_2b_2\mathbf{j}\times\mathbf{j} + a_2b_3\mathbf{j}\times\mathbf{k}\\
&\qquad + a_3b_1\mathbf{k}\times\mathbf{i} + a_3b_2\mathbf{k}\times\mathbf{j} + a_3b_3\mathbf{k}\times\mathbf{k}\\
&= a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i})\\
&= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}
\end{aligned}$$

2 If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then

$$\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle$$

In order to make this expression for $\mathbf{a}\times\mathbf{b}$ easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example, $\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

3 $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 3 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38$$

If we now rewrite (2) using second-order determinants and the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we see that the cross product of $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

4 $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

In view of the similarity between Equations 3 and 4, we often write

5 $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 5 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 3, we obtain Equation 4. The symbolic formula in Equation 5 is probably the easiest way of remembering and computing cross products.

EXAMPLE 3 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k}$$

$$= (-15 - 28)\,\mathbf{i} - (-5 - 8)\,\mathbf{j} + (7 - 6)\,\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$ ■

EXAMPLE 4 Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. We know from (9.2.1) that

$$\overrightarrow{PQ} = (-2 - 1)\mathbf{i} + (5 - 4)\mathbf{j} + (-1 - 6)\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1 - 1)\mathbf{i} + (-1 - 4)\mathbf{j} + (1 - 6)\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3 \rangle$, would also work. ■

EXAMPLE 5 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION In Example 4 we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of the cross product:

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$. ■

Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the **scalar triple product** of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Its geometric significance can be seen by considering the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (See Figure 7.) The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Thus, the volume of the parallelepiped is

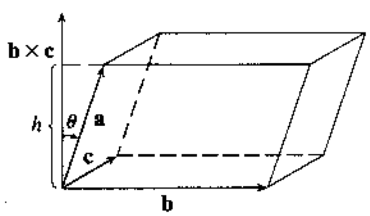


FIGURE 7

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Therefore, we have proved the following:

> The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product:
>
> $$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Instead of thinking of the parallelepiped as having its base parallelogram determined by $\mathbf{b}$ and $\mathbf{c}$, we can think of it with base parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. In this way, we see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

But the dot product is commutative, so we can write

6 $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

Suppose that **a**, **b**, and **c** are given in component form:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \qquad \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

Then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \right]$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

This shows that we can write the scalar triple product of **a**, **b**, and **c** as the determinant whose rows are the components of these vectors:

7 $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

EXAMPLE 6 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar; that is, they lie in the same plane.

SOLUTION We use Equation 7 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18) = 0$$

Therefore, the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar. ▮

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is called the **vector triple product** of **a**, **b**, and **c**. The proof of the following formula for the vector triple product is left as Exercise 30.

8 $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

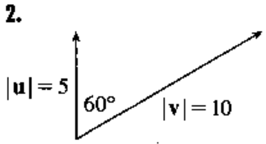
Formula 8 will be used to derive Kepler's First Law of planetary motion in Chapter 10.

9.4 Exercises

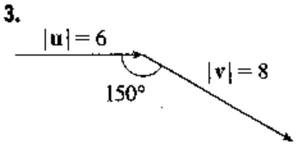
1. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.
   (a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ (b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$
   (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ (d) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$
   (e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ (f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

2–3 ■ Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.

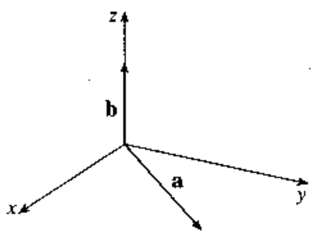
2.



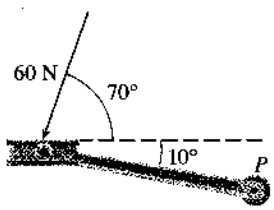
3.



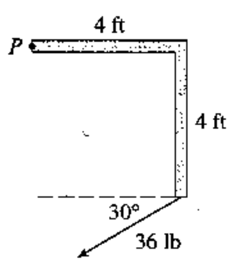
4. The figure shows a vector $\mathbf{a}$ in the xy-plane and a vector $\mathbf{b}$ in the direction of $\mathbf{k}$. Their lengths are $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.
   (a) Find $|\mathbf{a} \times \mathbf{b}|$.
   (b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.



5. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.



6. Find the magnitude of the torque about P if a 36-lb force is applied as shown.



7–11 ■ Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

7. $\mathbf{a} = \langle 1, -1, 0 \rangle$, $\mathbf{b} = \langle 3, 2, 1 \rangle$

8. $\mathbf{a} = \langle -3, 2, 2 \rangle$, $\mathbf{b} = \langle 6, 3, 1 \rangle$

9. $\mathbf{a} = \langle t, t^2, t^3 \rangle$, $\mathbf{b} = \langle 1, 2t, 3t^2 \rangle$

10. $\mathbf{a} = \mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$

11. $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

12. If $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. Sketch $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ as vectors starting at the origin.

13. Find two unit vectors orthogonal to both $\langle 1, -1, 1 \rangle$ and $\langle 0, 4, 4 \rangle$.

14. Find two unit vectors orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

15. Find the area of the parallelogram with vertices $A(-2, 1)$, $B(0, 4)$, $C(4, 2)$, and $D(2, -1)$.

16. Find the area of the parallelogram with vertices $K(1, 2, 3)$, $L(1, 3, 6)$, $M(3, 8, 6)$, and $N(3, 7, 3)$.

17–18 ■ (a) Find a vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

17. $P(1, 0, 0)$, $Q(0, 2, 0)$, $R(0, 0, 3)$

18. $P(2, 0, -3)$, $Q(3, 1, 0)$, $R(5, 2, 2)$

19. A wrench 30 cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $\langle 0, 3, -4 \rangle$ at the end of the wrench. Find the magnitude of the force needed to supply 100 J of torque to the bolt.

20. Let $\mathbf{v} = 5\mathbf{j}$ and let $\mathbf{u}$ be a vector with length 3 that starts at the origin and rotates in the xy-plane. Find the maximum and minimum values of the length of the vector $\mathbf{u} \times \mathbf{v}$. In what direction does $\mathbf{u} \times \mathbf{v}$ point?

21–22 ■ Find the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

21. $\mathbf{a} = \langle 6, 3, -1 \rangle$, $\mathbf{b} = \langle 0, 1, 2 \rangle$, $\mathbf{c} = \langle 4, -2, 5 \rangle$

22. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$, $\mathbf{c} = 2\mathbf{i} + 3\mathbf{k}$

23–24 ■ Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.

23. $P(1, 1, 1)$, $Q(2, 0, 3)$, $R(4, 1, 7)$, $S(3, -1, -2)$

24. $P(0, 1, 2)$, $Q(2, 4, 5)$, $R(-1, 0, 1)$, $S(6, -1, 4)$

25. Use the scalar triple product to verify that the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$, and $\mathbf{c} = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ are coplanar.

26. Use the scalar triple product to determine whether the points $P(1, 0, 1)$, $Q(2, 4, 6)$, $R(3, -1, 2)$, and $S(6, 2, 8)$ lie in the same plane.

27. (a) Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

28. (a) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

29. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

30. Prove the following formula for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

31. Use Exercise 30 to prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

32. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

33. Suppose that $\mathbf{a} \neq \mathbf{0}$.
(a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?
(c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

34. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

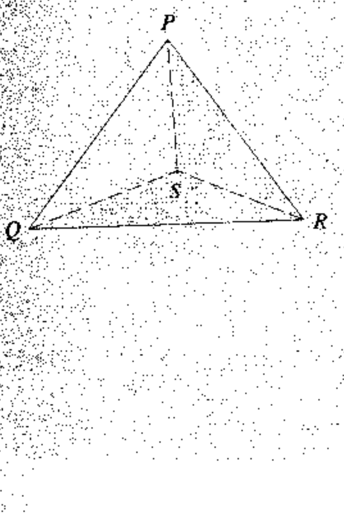
$$\mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

(These vectors occur in the study of crystallography. Vectors of the form $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$, where each n_i is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ form the *reciprocal lattice*.)
(a) Show that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i \neq j$.
(b) Show that $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.
(c) Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \dfrac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.

Discovery Project

The Geometry of a Tetrahedron



A tetrahedron is a solid with four vertices, P, Q, R, and S, and four triangular faces, as shown in the figure.

1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S, respectively, and directions perpendicular to the respective faces and pointing outward. Show that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

2. The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.
(a) Find a formula for the volume of a tetrahedron in terms of the coordinates of its vertices P, Q, R, and S.
(b) Find the volume of the tetrahedron whose vertices are $P(1, 1, 1)$, $Q(1, 2, 3)$, $R(1, 1, 2)$, and $S(3, -1, 2)$.