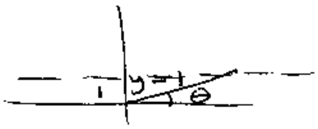
Answers to Fall 2000 Math 21a Final Exam

(1) switch to polar coordinates



$\frac{1}{r} = \sin\theta$, $\frac{1}{\sin\theta} \le r < \infty$, and $0 \le \theta \le \pi$

then $\int_{-\infty}^{\infty} \int_{1}^{\infty} y^{1/2}(x^2+y^2)^{-3/2}\, dy\, dx = \int_0^{\pi} \int_{1/\sin\theta}^{\infty} (r\sin\theta)^{1/2}\, r^{-3}\, r\, dr\, d\theta$

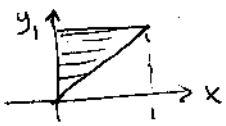
$$= \int_0^{\pi} \int_{1/\sin\theta}^{\infty} r^{-3/2} (\sin\theta)^{1/2}\, dr\, d\theta = \int_0^{\pi} (\sin\theta)^{1/2} \left[-2r^{-1/2}\right]\Big|_{\frac{1}{\sin\theta}}^{\infty} d\theta$$

(improper integrals → should use notation with limits to evaluate:)

$$= \lim_{h\to\infty} \int_0^{\pi} (\sin\theta)^{1/2} \left[-2r^{-1/2}\right]\Big|_{\frac{1}{\sin\theta}}^{h} d\theta$$

$$= \int_0^{\pi} 2\sin\theta\, d\theta = 4$$

(2) in xy plane:



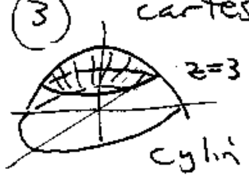
change order $0 \le y \le 1$
$0 \le x \le y$
$0 \le z \le \pi/2$

then get $\int_0^{\pi/2} \int_0^1 \int_0^y \cos(z)\sin(z)\sin(y^2\cos(z))\, dx\, dy\, dz$

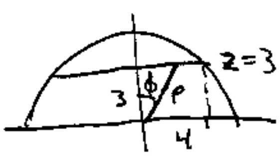
$$= \int_0^{\pi/2} \int_0^1 y\cos(z)\sin(z)\sin(y^2\cos(z))\, dy\, dz$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(-\cos(\cos z)\sin z + \sin z\right) dz = \frac{-\sin(1)+1}{2}$$

(3) cartesian: Volume $= \int_3^5 \int_0^{\sqrt{25-z^2}} \int_0^{\sqrt{25-z^2-y^2}} dx\, dy\, dz$



cylindrical: Volume $= \int_0^{\pi/2} \int_0^4 \int_3^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta$



spherical: Volume $= \int_0^{\pi/2} \int_0^{\tan^{-1}(4/3)} \int_{\frac{3}{\cos\phi}}^{5} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

(4) Find extremes on boundary: constraint function
$g(x,y) = x^2+y^2 = 25$, max/min $f(x,y) = x^2+y^2-4x-3y$

$\nabla g = \lambda \nabla f$, so $2x = \lambda(2x-4)$
$2y = \lambda(2y-3)$, so $\frac{2x}{2x-4} = \frac{2y}{2y-3}$

(note $x \neq 2$ and $y \neq \frac{3}{2}$, as the equations would read $4 = \lambda \cdot 0$ and $3 = \lambda \cdot 0$ respectively, impossible)

so $2x(2y-3) = 2y(2x-4)$

or $4xy - 6x = 4xy - 8y$, so $8y = 6x$, $y = \frac{3}{4}x$,

then $x^2+y^2 = x^2+\left(\frac{3}{4}x\right)^2 = x^2\left(1+\frac{9}{16}\right) = 25$, $\frac{25}{16}x^2 = 25$

so $x = \pm 4$, $y = \pm 3$

check these two points: $f(4,3) = 0$ $f(-4,-3) = 50$.

Next check interior of $x^2+y^2 \leq 25$:

$\nabla f = (2x-4, 2y-3)$ so $x = 2$, $y = \frac{3}{2}$

$f(2, \frac{3}{2}) = -\frac{25}{4}$

so global min at $(2, 3/2)$, global max at $(-4,-3)$

(5) distance from (x,y,z) to $(1,1,0)$ is $\sqrt{(x-1)^2+(y-1)^2+z^2}$
minimize $(\text{distance})^2 = (x-1)^2+(y-1)^2+z^2 = f(x,y,z)$
subject to constraint function $g(x,y,z) = z^2-x^2-y^2 = 1$

so $\nabla f = \langle 2(x-1), 2(y-1), 2z\rangle = \lambda \nabla g = \lambda\langle -2x, -2y, 2z\rangle$

so $2x-2 = -2x\lambda$ $z \neq 0$, since then $-x^2-y^2 = 1$ impossible,
$2y-2 = -2y\lambda$ so $\lambda = 1$, then
$2z = \lambda 2z$ $2x-2 = -2x$ so $4x = 2$, $x = \frac{1}{2}$
$2y-2 = -2y$ so likewise $y = 1/2$

closest point is then $\left(\frac{1}{2}, \frac{1}{2}, \sqrt{1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right)$
$= \left(\frac{1}{2}, \frac{1}{2}, \sqrt{\frac{3}{2}}\right)$ (since $z \geq 0$)

(6) approximation: $L = f(0,0,0) + f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0)$

$f_x = e^{-z^2\sin(y)} + 2x\cos(x^2+z)$ at $(0,0,0)$ $f_x(0,0,0) = 1$

$f_y = -x\,e^{-z^2\sin(y)}(\cos y)z^2$ at $(0,0,0)$ $f_y(0,0,0) = 0$

$f_z = -2z(\sin y)x\,e^{-z^2\sin(y)} + \cos(x^2+z)$ and $f_z(0,0,0) = 1$

so $L = 0 + x + 0 + z = x + z$.

so when $x = .001$, $y = .002$, $z = .003$

Linear approximation $L = .001 + .003 = .004 = \frac{1}{250}$ (d)

(7) (a) normal vector to S: $f(x,y,z) = \left(\sqrt{x^2+y^2}-4\right)^2 + z^2 = 1$

$\nabla f = \left\langle 2\left(\sqrt{x^2+y^2}-4\right)\left(\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x\right),\ 2\left(\sqrt{x^2+y^2}-4\right)\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y,\ 2z\right\rangle$

so $\nabla f(3,4,0) = \langle \frac{3}{5}, \frac{4}{5}, 0\rangle$, which is unit length already.

so $\langle \frac{3}{5}, \frac{4}{5}, 0\rangle \cdot \langle 5,0,0\rangle = 3$

(b) at $(1,\sqrt{3},1)$ $\nabla f(1,\sqrt{3},1) = \langle -2, -2\sqrt{3}, 2\rangle$, unit length vector in same direction: $\frac{1}{\sqrt{20}}\langle -2,-2\sqrt{3},2\rangle$

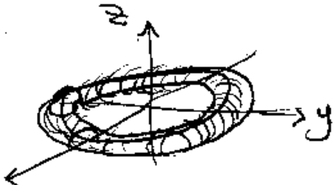
cross product with $\langle \sqrt{3}, 1, 0\rangle = \frac{1}{\sqrt{20}}\langle 2, -2\sqrt{3}, -4\rangle$

norm $= \sqrt{\frac{8}{5}}$

(c) if $z=0$, then $\left(\sqrt{x^2+y^2}-4\right) = \pm 1$, largest value for x when $y=0$, $\sqrt{x^2+y^2} = \pm x = -1+4$ or $-1-4 = -5$

so maximum value for x is $+5$

(d) S is simply a torus, or doughnut



max. value of x is $+5$, minimum at other end of doughnut along x-axis is -5, so the function has two extremes

(e) None, furthest point away is on outside of doughnut, where $\sqrt{x^2+y^2} = 5$, a distance of 5 from the origin

(9) (a) parametrize S by usual $\langle x, y, f(x,y)\rangle = \langle x, y, 1-x^2-y^2\rangle$ for $x^2+y^2 \le 1$

Then $(\vec{r}_x \times \vec{r}_y) = \langle 2x, 2y, 1\rangle$

here $\vec{F} = \langle x + y\sin((1-x^2-y^2)^2),\ y + x\sin((1-x^2-y^2)^2),\ 1-2(1-x^2-y^2)\rangle$

so $\iint_S \vec{F}\cdot d\vec{S} = \iint_R \vec{F}\cdot\langle 2x, 2y, 1\rangle\, dA$ where R is the circle $x^2+0^2 \le 1$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(2x^2 + 4xy\sin((1-x^2-y^2)^2) + 2y^2 + 1 - 2(1-x^2-y^2)\right) dy\, dx$$

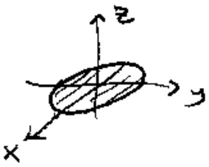
(b) calculate $\operatorname{div}\vec{F} = 1+1-2 = 0$, so $\iint_{S''} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div} F\, dV = 0$

if S'' is the closed surface consisting of S from part (a) as its top and the circle $x^2+y^2\le 1$ (with $z=0$) as its bottom, S'

so $0 = \iint_{S''} \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S} + \iint_{S'} \vec{F}\cdot d\vec{S}$, so the flux integral over the circle $x^2+y^2\le 1$ $z=0$ gives the same answer (up to $+/-$)

(c) We use the observation in (b):

for S': $x^2+y^2\le 1$ $z=0$ $\vec{F} = \langle x, y, 1\rangle$ (nice simplification!)



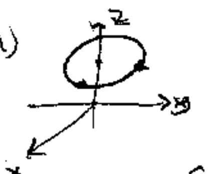
S' is parametrized as just $\langle x, y, 0\rangle$

so $\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1\rangle$

and $\iint_{S'} \vec{F}\cdot d\vec{S} = \iint_R \langle x, y, 1\rangle\cdot\langle 0,0,1\rangle\, dx\,dy = \iint_R dx\,dy$

over R, the circular region $x^2+y^2\le 1$, but this now is just the area of said circle $= \pi$

(10) (a)



unit circle at height $z = \sqrt{\frac{\pi}{2}}$, parametrize path as

$x = \cos\theta$, $y = \sin\theta$, $z = \sqrt{\frac{\pi}{2}}$ for $0 \le \theta \le \pi/4$

so $\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/4} \vec{F}\left(\cos\theta, \sin\theta, \sqrt{\tfrac{\pi}{2}}\right)\cdot\langle -\sin\theta, \cos\theta, 0\rangle\, d\theta$

$$= \int_0^{\pi/4} \langle \cos\theta + \sin\theta, \sin\theta + \cos\theta, 1 - 2\sqrt{\tfrac{\pi}{2}}\rangle\cdot\langle -\sin\theta, \cos\theta, 0\rangle\, d\theta$$

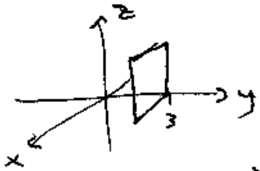
$$= \int_0^{\pi/4} (\cos^2\theta - \sin^2\theta)\, d\theta = \int_0^{\pi/4} \cos(2\theta)\, d\theta = \frac{1}{2}$$

(10) (b) all such closed path loops would lie parallel to xy plane (for the tangent vectors to be perpendicular to z-axis), so $\vec{F} \cdot d\vec{r}$ wouldn't depend on the z component of either $\vec{F}$ or $\vec{r}(t)$, so it could be reduced to a two dimensional problem (also note $\vec{r}'(t) = \langle *, *, 0 \rangle$, so $\vec{F} \cdot \vec{r}'(t)$ just has non-zero components in first two places)

Now consider these first two components

$$\vec{F} = \langle \underbrace{x + y\sin(z^2)}_{P},\ \underbrace{y + x\sin(z^2)}_{Q},\ \cdots \rangle$$

but $\frac{\partial Q}{\partial x} = \sin(z^2) = \frac{\partial P}{\partial y}$, thus in any plane with z=constant this two dimensional vector field $\langle x + y\sin(z^2), y + x\sin(z^2) \rangle$ is conservative, and so line integral around closed path is 0

(c) 

use Stokes' Theorem turn line integral around boundary of rectangle into double integral over interior of rectangle . . .

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl } \vec{F}) \cdot dS$$

R is parametrized simply by $\langle x, 3, z \rangle$

so $(\vec{r}_x \times \vec{r}_z) = \langle 0, -1, 0 \rangle$

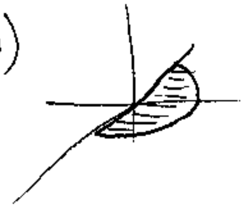
and integral becomes $\displaystyle\int_0^1 \int_0^{\sqrt{\pi}} (\text{curl } \vec{F}) \cdot \langle 0, -1, 0 \rangle \, dz \, dx$

so just need to find middle component of curl $\vec{F} = \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)$

$= 2yz\cos(z^2)$ (when $y = 3$)

so integral becomes $\displaystyle\int_0^1 \int_0^{\sqrt{\pi}} 2 \cdot 3 \cdot z\cos(z^2) \, dz \, dx$

$$= \int_0^1 \left(3\sin(z^2)\right)\Big|_0^{\sqrt{\pi}} dx = 0$$

(11) 

when $z=0$ surface is $x^4+2y^2x^2+y^4=16$

or $(x^2+y^2)^2 = 4^2$

i.e. just circle $x^2+y^2=4$

calculate surface area of graph of $z^4 = 16-(x^2+y^2)^2$

easiest parametrization : $\langle x, y, (16-(x^2+y^2)^2)^{1/4} \rangle$

(i.e. $\langle x, y, f(x,y) \rangle$)

then $|\vec{r}_x \times \vec{r}_y| = \sqrt{1+(f_x)^2+(f_y)^2}$

$$f_x = \frac{1}{4}\left(16-(x^2+y^2)^2\right)^{-3/4}\left(-2(x^2+y^2)\right)\cdot(2x)$$

and $$f_y = \frac{1}{4}\left(16-(x^2+y^2)^2\right)^{-3/4}\left(-2(x^2+y^2)\right)\cdot(2y)$$

so $$\text{Surface area} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+(f_x)^2+(f_y)^2}\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{(x^2+y^2)^3\left(16-(x^2+y^2)^2\right)^{-3/2}+1}\,dy\,dx$$

(12) (a) True for some, not all

(b) True for all

(c) Never true

(d) True for all

(e) True for some, not all

PART C: The questions in this part of the exam are only for students in the BioChem sections. Students in either the Regular or Physics sections do not answer questions in this part of the exam.

9. According to one source, the sensitivity of the SUDS test for HIV is 0.999 and the specificity of this test is 0.996. In the nation of Botswana, about one in three adults is infected with HIV.
   a) What are the positive and negative predictive values of the SUDS test for the nation of Botswana?
   b) The sensitivity and positive predictive value of a test represent information that medical practitioners need to know. In practice, which of these two can actually be measured? Explain your reasoning in a brief sentence.
   c) Would a medical practitioner rather know the sensitivity or positive predictive value of an HIV test? Explain your reasoning in a brief sentence.

A= SUDS test positive
B = infected with HIV

sensitivity = $Pr(A|B) = 0.999$ $\quad$ $Pr(\bar{A}|B) = 1 - Pr(A|B) = 0.001$

specificity = $Pr(\bar{A}|\bar{B}) = 0.996$ $\quad$ $Pr(A|\bar{B}) = 1 - Pr(\bar{A}|\bar{B}) = 0.004$

$Pr(B) = \frac{1}{3}$ $\quad$ $Pr(\bar{B}) = 1 - Pr(B) = \frac{2}{3}$

a) $$PV^+ = Pr(B|A) = \frac{Pr(A|B)Pr(B)}{Pr(A|B)Pr(B) + Pr(A|\bar{B})Pr(\bar{B})} = \frac{0.999\times\frac{1}{3}}{0.999\times\frac{1}{3} + 0.004\times\frac{2}{3}} = \underline{\underline{\frac{999}{1007}}}$$

$$PV^- = Pr(\bar{B}|\bar{A}) = \frac{Pr(\bar{A}|\bar{B})Pr(\bar{B})}{Pr(\bar{A}|\bar{B})Pr(\bar{B}) + Pr(\bar{A}|B)Pr(B)} = \frac{0.996\times\frac{2}{3}}{0.996\times\frac{2}{3} + 0.001\times\frac{1}{3}} = \underline{\underline{\frac{1992}{1993}}}$$

b) Sensitivity can be measured. We find a group of adults who we know are infected with HIV and give them SUDS tests.

c) Positive predictive value. The higher the predictive value of an HIV test, the more valuable the test — if $PV^+ = 1$, then a medical practitioner know for sure that a person who tests positive must be infected with HIV.

10. A quarter of the diners at a certain restaurant order a fish dinner, half order a meat dinner and the remainder order only a green salad. Two thirds of those that order a fish dinner also order a green salad as a first course, while one third of those that order a meat dinner order a green salad as as a first course.

a) What is the probability that a diner at this restaurant will order a salad?
b) Suppose that you see a diner eating a green salad. What is the probability that th diner will only have a salad? What is the probability that the diner will also have fish?
c) If the average menu price for a fish dinner is \$8, the average for a meat dinner is \$10 and that for a green salad is \$2, what is the average amount (excluding tip and tax) spent by a diner at this restaurant?

F = order fish , M = order meat $\quad F \cap M = \phi$

O = order only green salad $= \overline{F \cup M}$

S = order salad

$Pr(F) = \frac{1}{4}$, $Pr(M) = \frac{1}{2}$, $Pr(O) = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$

$Pr(S|F) = \frac{2}{3}$, $Pr(S|M) = \frac{1}{3}$

a) F, M, O mutually exclusive, exhaustive

$$Pr(S) = Pr(S|F)Pr(F) + Pr(S|M)Pr(M) + Pr(S|O)Pr(O)$$
$$= \frac{2}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{2} + 1 \times \frac{1}{4} = \frac{1}{6} + \frac{1}{6} + \frac{1}{4} = \underline{\underline{\frac{7}{12}}}$$

b) $$Pr(O|S) = \frac{Pr(S|O)Pr(O)}{Pr(S)} = \frac{1 \cdot \frac{1}{4}}{\frac{7}{12}} = \underline{\underline{\frac{3}{7}}}$$

$$Pr(F|S) = \frac{Pr(S|F)Pr(F)}{Pr(S)} = \frac{\frac{2}{3} \cdot \frac{1}{4}}{\frac{7}{12}} = \underline{\underline{\frac{2}{7}}}$$

c) $$8 \times Pr(F) + 10 \times Pr(M) + 2 \times Pr(S)$$
$$= 8 \times \frac{1}{4} + 10 \times \frac{1}{2} + 2 \times \frac{7}{12} = \underline{\underline{8\frac{1}{6}}}$$

11. On average, three phone calls in four to a computer help hot line receive busy signals. Assume that this statistic is independent of the number of times that any given person calls the provider. Also, suppose that the help hotline receives an average of 1000 calls per hour, and that the number of calls received in any given hour has a Poisson distribution.
   a) Write down an expression that gives the probability that your first ten calls to the help hotline get busy signals and your eleventh call gets through. Do not evaluate your expression.
   b) Write down an expression that gives the probability that at least your first ten calls to the help hotline get busy signals. Do not evaluate your expression.
   c) Write down, but do not evaluate, an expression for the probability that the hot line receives 3000 calls in a given two hour period.

   Do not justify your answer.

a) probability of one call to the help hotline get busy signals is $\frac{3}{4}$

$$\underline{\underline{\left(\frac{3}{4}\right)^{10} \cdot \frac{1}{4}}}$$

b) $\underline{\underline{\left(\frac{3}{4}\right)^{10}}}$

c) X = number of calls in a two hour period

$X \sim \mathrm{POI}(1000 \times 2) = \mathrm{POI}(2000)$

$$\Pr(X = 3000) = \underline{\underline{\frac{e^{-2000}(2000)^{3000}}{3000!}}}$$

12. Suppose that the following probabilities are normally distributed with the indicated mean, μ, and standard deviation σ:

i) The average daily Boston temperature in January: $\mu = -2°$ C and $\sigma = 4°$ C.

ii) The average daily Boston temperature in July: $\mu = 25°$ C and $\sigma = 8°$ C.

iii) The average daily Boston temperature in May: $\mu = 15°$ C and $\sigma = 6°$ C.

Based on this data, re-order the following list so that the probability of occurrence decreases going down the list. Do not justify your answer.

a) The average temperature on July 15 is greater than or equal to 32° C.

b) The average temperature on January 8 is less than – 5° C.

c) The average temperature on May 15 is greater than 20° C.

d) The average temperatures on July 15 is between 25° C and 33° C.

X_1 = average daily Boston temperature in January

X_7 " July

X_5 " May

$X_1 = -2 + 4Z$, $X_7 = 25 + 8Z$, $X_5 = 15 + 6Z$, where $Z \sim N(0,1)$

a) $Pr(X_7 \geq 32) = Pr(25 + 8Z \geq 32) = Pr(Z \geq \frac{7}{8})$

b) $Pr(X_1 < -5) = Pr(-2 + 4Z < -5) = Pr(Z < \frac{-3}{4}) = Pr(Z > \frac{3}{4})$

c) $Pr(X_5 > 20) = Pr(15 + 6Z > 20) = Pr(Z > \frac{5}{6})$

d) $Pr(25 < X_7 < 33) = Pr(25 < 25 + 8Z < 33) = Pr(0 < Z < 1)$

$\underline{\underline{d > b > c > a}}$