

Math 21a Practice Final Exam

What follows is a model of the final exam (it is more or less the exam given in Fall 1999 in this course). The upcoming final exam will be roughly similar to this one in length and difficulty. To study for the upcoming exam, you will benefit from reworking the suggested problems and weekly assignments from the course. These and the answers are either in the text books or posted on the course website. You should also work the practice problems for final exam supplied elsewhere at this website. The answers to the latter are also supplied at this website. After you have worked a large number of such practice problems and checked your answers, try taking this exam as practice for the real thing. In this regard, note that you will have three hours for the real thing. The answers to this practice final exam are provided elsewhere at the website.

By the way, in the real exam, each problem will be printed on a separate page and you will be asked to provide your answers on that page and to use the back of that page if you need more space to show your work.

Remember that the exam on *Tuesday, January 22* is from *9:15am to 12:15 pm*. People with last names alphabetized starting with *A through Foley* take their exam in *Science Center Lecture Hall A*, and those with names alphabetized starting with *Fr through Z* take the exam in *Science Center Lecture Hall B*. Please come a few minutes early as the exam will be started promptly at 9:15 by the university's proctors.

1) __ 2) __ 3) __ 4) __ 5) __ 6) __ 7) __ 8) __ 9) __ 10) __ 11) __ 12) __ : Total _____

Name: _____

Circle the name of your Section TA:

Arinkin(10)•Arinkin(12)•Bamberg•Cornut•Kaplan•Karu•Knill•Libine•Liu•Taubes•Williams

Instructions:

- Print your name in the line above and circle the name of your section TF.

❖ **THERE ARE FOUR PARTS TO THIS EXAM:**

A. All students answer Part A questions.

B. Only students in either the Regular or Physics sections answer Part B questions.

C. Only students in the BioChem sections answer Part C questions.

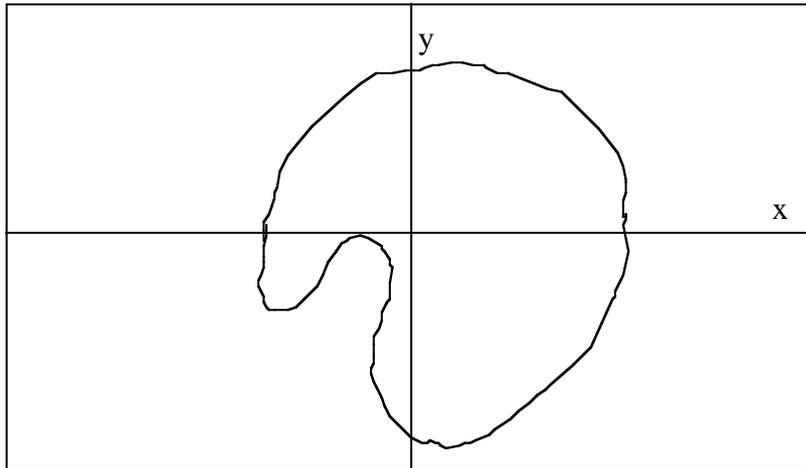
D. Only students in the Computer Science Section answer Part D questions.

- Answer each of the questions below in the space provided. If more space is needed, use the back of the facing page.
- Please give justification for answers if you are not told otherwise.
- Please write neatly. Answers which are deemed illegible by the grader will not receive credit.
- No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students. Use only your brain and a pencil.
- All problems will count the same amount.
- Neither unstaple nor remove pages from your exam booklet.
- Note that vectors are indicated below by bold face type.

In agreeing to take this exam, you are implicitly agreeing to act with fairness and honesty.

PART A: The questions in this part of the exam should be answered by all students.

1. Pictured below is a curve in the x - y plane on which a flea lives. The temperature of the plane on a region containing the curve is $T(x, y) = 20 + x + y$.



- Mark all of the points on the curve where the curve's temperature is clearly a local maximum.
 - Mark all of the points on the curve where the curve's temperature is clearly a local minimum.
 - If the flea travels counter clockwise around the curve at constant speed, label all of the points on the curve where the temperature is increasing at the fastest rate.
 - For the same flea as above, label all of the points on the curve where the temperature is decreasing at the fastest rate.
 - Mark all points on the curve which could plausibly have zero rate of change of the temperature without being a local maximum or minimum.
2. First use Cartesian coordinates, then use cylindrical coordinates, and finally use spherical coordinates to write an iterated triple integral expressions for the integral of xyz over the region in space where $0 \leq z \leq (1 - x^2 - y^2)^{1/2}$, $x^2 + y^2 \leq 1$ and $x - y \geq 0$ and $y \geq 0$. Do not evaluate your expressions.

3. Suppose that $f(x, y, z)$ is a function on \mathbb{R}^3 whose value at the origin is 6. Suppose that at the origin, the dot product of ∇f with $(1, 3, 1)$ equals 1, the dot product of ∇f with $(2, 1, 3)$ equals 2, and with $(1, 2, 3)$ equals 3. Write down:
- A non-zero vector which is tangent at the origin to the level set $f = 6$.
 - The equation for the tangent plane at the origin to the level set $f = 6$.
 - The linear approximation to f at the origin.
 - The directional derivative at the origin of f in the direction of $\frac{1}{\sqrt{3}}(1, 1, 1)$.
4. Integrate the function xyz over region where $x \geq 0$, $y \geq 0$, $z \geq 0$ and $z + x^2 + y^2 \leq 1$.
5. Find all points in the region $x^2 + y^2 + z^2 \leq 1$ at which the gradient of $\frac{1}{16}x^2 + \frac{1}{4}y^2 + z^2$ has largest length (that is, largest magnitude).
6. Give a parametric equation for the line which lies in the planes $x + y - z = 1$ and $x + 2y + z = 1$.
7. Integrate the function $\cos(\theta)$ over region of the plane which is given in polar coordinates, by the condition that $\cos^2(\theta) \leq r \leq \cos(\theta)$.
8. Suppose that a function u obeys the equation

$$u_{xx} + u_{yy} = 1$$

where $x^2 + y^2 \leq 1$. Note that there are infinitely many functions which obey this equation. Here are two examples: $u = \frac{1}{4}(x^2 + y^2)$ and $u = \frac{1}{6}x^2 + \frac{1}{3}y^2 - \frac{1}{27}x$.

Label each of the following statements with 'A' if the statement is true for every possible solution, 'S' if it is true for some but not all solutions, or 'N' if the statement is not true for any solution. Justify your answer in each case.

- The gradient of u must be non-zero at some point.
- The function u is independent of x .
- The function u achieves its global maximum where $x^2 + y^2 = 1$.
- The function u obeys $u(0) > 0$ and $u < 0$ where $x^2 + y^2 > \frac{1}{2}$.
- The function u achieves its global minimum where $x^2 + y^2 = 1$.

PART B: The questions in this part of the exam are only for students in either the Regular or the Physics sections. Students in the BioChem or Computer Science sections do not answer questions in this part of the exam; students in the BioChem sections turn immediately to Part C of the exam and those in the CS section turn immediately to Part D.

9. Let $\mathbf{F} = (x + xz, y - yz, z^2)$. Here are two surfaces in \mathbb{R}^3 with the same boundary:

A: $z = (1 - x^2 - y^2)^{1/2}$ with $x^2 + y^2 \leq 1$.

B: $z = 2(1 - x^2 - y^2)^{1/2}$ with $x^2 + y^2 \leq 1$.

- Label the surfaces with 1 or 2; where 1 has the lowest flux of \mathbf{F} and 2 the highest. Ties are allowed. Use the normal which has positive dot product with $(0, 0, 1)$. Justify your answer.
- Label the surfaces with 1 or 2; where 1 has the lowest flux of $\text{curl}(\mathbf{F})$ and 2 has the highest. Ties are allowed here too. Use the same normal as in Part a. Justify your answer.
- Compute the flux of $\text{curl}(\mathbf{F})$ through the surface where $z = 1 - x^2 - y^2$ with $x^2 + y^2 \leq 1$.

10. a) Write down a 2-variable, iterated integral for the area of the portion of the surface where $x^2 - 4(y^2 + z^2) = 0$ and $0 \leq x \leq 1$. Don't evaluate your integral.

b) Write down a 2-variable, iterated integral for the flux of $\mathbf{F} = (x, 0, 0)$ through the surface as defined using the normal vector whose dot product with $(1, 0, 0)$ is negative.

11. In each case below, either write down a vector field on \mathbb{R}^3 with the desired properties, or else explain why no such vector field exists.

- The curl is $(1, 1, 0)$ and the flux is zero through any surface in a plane where z is constant.
- The curl is $(1, 1, 0)$ and the line integral is zero around all closed loops in the $x = 0$ plane.
- The divergence is 2 and the line integral is zero around all closed loops.
- The divergence is 2 and the flux through any closed surface is zero.

12. a) Write down a vector field on \mathbb{R}^2 whose line integral around the counter-clockwise oriented boundary of every bounded region in the plane gives the integral of x^2 over that region.

b) Parametrize the circle where $x^2 + y^2 = 1$ and directly compute the line integral of your vector field around this circle.

PART C: The questions in this part of the exam are only for students in the BioChem sections. Students in the Regular, Physics or Computer Science sections do not answer questions in this part of the exam. Students in the Computer Science section turn immediately to Part D of this exam. Regular and Physics section students should do Part B instead.

9. Guildenstern flips a biased coin that lands heads 70% of the time. He makes 80 flips.
- What are the expected number of heads and the standard deviation?
 - Give an exact expression for the probability that there are 62 or more heads in this experiment. Don't compute the numerical value of your expression.
 - Write an integral using a normal distribution whose value gives an approximate answer for the probability in Part b. Don't evaluate the integral.
10. Consider an experiment which consists of throwing three dice, each with sides numbered 1-6.
- What is the probability of getting more than 2 on at least one of the dice?
 - What is the probability that the sum showing on all three dice is less than or equal to 4?
 - Given that the first die shows 3, what is the probability that the sum of all three dice is 7?
11. In the state of Massachusetts, the mortality rate due to a certain rare cancer is 25 persons in a four year period (as measured over a long period of years).
- Use a Poisson distribution to give an expression for the probability of k deaths due to this cancer in a two year period. Don't compute the numerical value of your expression.
 - If 40 deaths are reported in a given four year period, how many standard deviations is this from the mean of the Poisson distribution that you used to answer Part a.
 - Using your Poisson distribution, give an expression (but don't evaluate it) for the probability of seeing 40 deaths in both the 4-year period that started in 1992 and in the 4-year period that started in 1996.
12. A group of people were tested to study the relationship between heart disease and physical fitness. Here is the data showing the relationship between exercise heart rate and heart disease:

<u>Heart disease (per 100 people)</u>	<u>Heart rate</u>
9	≤ 105
8	106-115
12	116-126
13	≥ 127

Suppose that the percentages of the overall population in each category are 20%, 30%, 40% and 10%, respectively. Let the positive test for heart disease be a heart rate of 127 or more and a negative test a rate of 126 or less.

- a) What is the overall rate of heart disease in the population?
- b) Compute the sensitivity and the predictive-value-positive (PV^+) of the test.
- c) Given a negative test, what is the probability that a person has heart disease?

PART D: The questions in this part of the exam are only for students in the Computer Science section. Students in the Regular, Physics or BioChem sections do not answer questions in this part of the exam. Students in Regular and Physics section students should do Part B instead and those in the BioChem sections should do Part C.

9. A police chief, to her amazement, has captured a famous con artist and story teller named Bernoulli. Bernoulli offers to tell a story every day on the condition that she not be jailed until the day that her story fails to make the chief laugh. The chief agrees to this proposal. Moreover, since the chief has a fine sense of humor, the probability that a story will NOT make the chief laugh is $p = 1/5$.
 - a) What is the probability that Bernoulli goes to jail after telling precisely 3 stories?
 - b) What is the probability that Bernoulli goes to jail on or before the third day?
 - c) On the day of her capture, the press asks "What day is Bernoulli most likely to go to jail?" What is the correct answer? Explain.
 - d) Given that Bernoulli is in jail by the end of the fourth day, what is the probability that she went to jail on the third day?
10. Suppose that a function G satisfies the equation $G(x+y) = G(x)G(y)$ for all positive numbers x and y . Prove by induction that $G(nx) = G(x)^n$ for any positive x and positive integer n .
11. Suppose that you want to write a computer program that shows that the collection of all two element sets of distinct positive integers is countable by listing the sets forever. Thus, its output might start with
 - 1 \rightarrow {1, 2},
 - 2 \rightarrow {1, 3},
 - ...

- a) What two requirements must be met by your program to justify a claim that it shows the countability of the collection of two element subsets of the positive integers?
- b) Provide an algorithm for listing sets that demonstrates the countability of the collection of all two element subsets of the positive integers and list its first six output lines.
- c) Assume the truth of the claim that a countable union of finite sets is countable and use it to prove the following assertion: *For any positive integer n, the collection of n element subsets of distinct, positive integers is countable.*

12. A two-dimensional random variable (X, Y) has a probability density function of two variables given by

$$\begin{aligned} f(x,y) &= 0 && \text{if either } x < 0, y < 0, x > 3 \text{ or } y > 2. \\ f(x,y) &= cx^2 y && \text{otherwise.} \end{aligned}$$

- a. Determine what value the constant c must have.
- b. Calculate the probability of the event $X^2 + Y^2 \leq 1$.
- c. Calculate the expectation of $X + Y$.