Last Name: ______________________________

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Math 21a Final Exam: Tuesday, May 22, 2001
(Regular and Physics Sections)

SECTION (CIRCLE ONE):

Robert Winters Michael Simonetti (CA) MWF 10-11	**Robert Winters** Lisa Carlivati (CA) MWF 11-12	**Paul Bamberg** Geoff Werner-Allen (CA) MWF 11-12 (Physics)
Dima Arinkin Nathan Moore (CA) MWF 12-1 (BioChem)	**Mark Lucianovic** Bryden Cais (CA) TTh 10-11:30	**Yang Liu** Anthony Varilly (CA) TTh 11:30-1

Question	Points	Score
1	8	
2	10	
3	8	
4	9	
5	8	
6	9	
7	9	
8	9	
9	12	
10	9	
11	9	
Total	100	

The time allotted for this exam is 3 hours.

Justify your answers carefully. No partial credit can be given for unsubstantiated answers.

If more space is needed, use the back of the previous page and make note of this.

Please write neatly. Answers which are deemed illegible by the grader will not receive credit.

No calculators, computers, or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students.

(1) An isosceles triangle has angle θ between its two sides of equal length. The midpoints of these two sides are joined to create an isosceles trapezoid. The diagonals of this isosceles trapezoid are perpendicular. By considering the diagonals of the trapezoid as vectors and taking their scalar product. determine the value of $\cos \theta$.

(2) The graph of $f(x,y,z) = xz^3 + yz^2 + x^2y = 18$ is a surface that includes the point $(x, y, z) = (1, 2, 2)$.

(a) Find a vector normal to this surface at the point (1, 2, 2).

(b) Suppose we were to move slightly radially outward in $\mathbf{R}^3$ from the point (1, 2, 2). At what rate would the function *f* change (per distance traveled) if we were to move in this way?

(c) Near the point (1, 2, 2), the equation $f(x,y,z) = 18$ defines z implicitly as a function of x and y. Find the numerical values of the partial derivatives of this function $z(x, y)$ at $(x, y) = (1, 2)$.

(d) Using a linear approximation, estimate the value of $z(1.1, 2.04)$.

(3) Find the maximum value of the function $f(x, y, z) = xy^3z^4$ for points (x, y, z) lying on the plane $x + y + z = 4$ such that $x \geq 0, y \geq 0, z \geq 0$. Justify your answer.

(4) Suppose that a student member of a Harvard committee has determined that the well-being of an employee is described by the function $W(s, p) = s^{0.7} p^{0.3}$, where s is the hourly salary and p is the number of hours per week spent in educational programs. The Harvard Corporation has decreed that the committee may only recommend values of s and p that satisfy the constraint $7s + 12p = 120$. Use the Method of Lagrange Multipliers to find the values of s and p that maximize employee well-being according to this model.

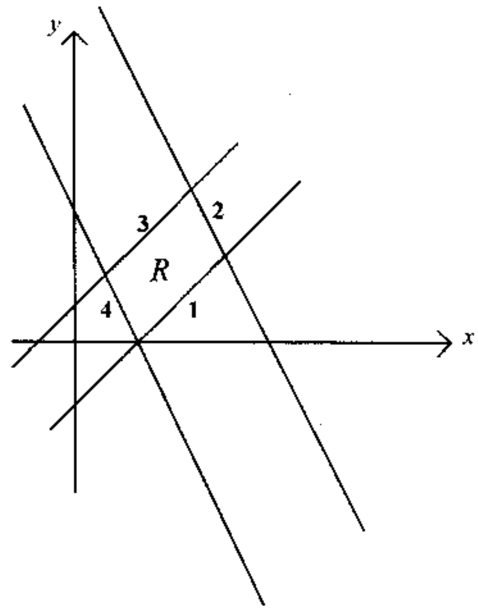
(5) The temperature T on a metal disc can be regarded either as a function of Cartesian coordinates x and y or of polar coordinates r and θ. The relationship between the two sets of coordinates is of course $x = r\cos\theta$, $y = r\sin\theta$.

(a) Using the Chain Rule, express $\frac{\partial T}{\partial \theta}$ in terms of r, θ, and the partial derivatives of T with respect to x and y.

(b) The partial derivative $\frac{\partial T}{\partial x}$ can also be regarded either as a function of r and θ or as a function of x and y. Express its partial derivative $\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right)$ in terms of r and θ and the second partial derivatives of T with respect to x and y.

(c) The temperature T, viewed as a function of r and θ, has the property that $\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}=0$. Using the Chain Rule, show that this property can also be expressed as $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$.

(6) Consider the region R in the xy-plane bounded by the lines
$y = -2x + 4$ (labeled **4** in diagram),
$y = -2x + 10$ (labeled **2** in diagram),
$y = x - 2$ (labeled **1** in diagram), and
$y = x + 1$ (labeled **3** in diagram).



Let $f(x, y) = 2x^2 - xy - y^2$ be defined for points (x, y) in the region R.

(a) Define new coordinates (u, v) such that in these new coordinates the region R is transformed into a rectangle with sides parallel to the u and v axes.
Give expressions for $u(x, y)$ and $v(x, y)$ and use these to solve for $x(u, v)$ and $y(u, v)$.

(b) Write down an expression for the average value of the function f over the region R, i.e. as a quotient of two integrals.

(c) Calculate these integrals using the coordinates you found in part (a).

(7) Find the numerical value of the integral $\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r^3 z^2\,dz\,dr\,d\theta$ by changing it from an integral expressed in cylindrical coordinates to an integral expressed in spherical coordinates.
[*Note: Though it's possible to calculate this integral in cylindrical coordinates, we do want it done in spherical coordinates.*]

(8) The intersection of the ellipsoid $4x^2 + y^2 + z^2 = 25$ with the plane $x = -2$ is a circle. Calculate the line integral of the vector field $\mathbf{F} = (xy - x, xz, x^2y)$ around this circle. Traverse this circle in the counterclockwise direction as viewed from the origin.

(9) Let S be the sphere of radius $\sqrt{2}$ centered about the origin and having the equation $x^2 + y^2 + z^2 = 2$.

(a) Find the surface area of the "drum" which is part of this sphere with $-1 \le z \le +1$, together with its top and bottom.

(b) Find the flux of the vector field $\mathbf{F} = (x, y, 0)$ outward through the side of the drum.

(problem continued on next page)

(9c) Find the flux of $\mathbf{G} = (0.\ 0, z)$ outward through the entire surface of the drum.

(9d) Based on your previous answers, find the flux of $\mathbf{H} = (x, y, z)$ outward through the entire surface of the drum.

(9e) What is the volume of the drum?

(10) Use Stokes' Theorem to evaluate the line integral of the vector field $\mathbf{F} = (x^3 - 2y, 3xz + y^3, yz)$ along the curve of intersection of the paraboloid $z = x^2 + y^2 + 2x + 2y - 9$ and the plane $z = 2x + 2y$. Follow this curve counterclockwise as viewed looking down from the positive z-axis.

(11) Find the net flux of the vector field $\mathbf{F} = (x^3 - 2y, 3xz + y^3, yz)$ outward through the closed surface bounded below by the paraboloid $z = x^2 + y^2 + 2x + 2y - 9$ and above by the plane $z = 2x + 2y$.
[*Note: These are the same surfaces and the same vector field as in the previous problem.*]