Last Name: Answers

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Math 21a Exam #2: Tuesday, April 4, 2000

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Question	Points	Score
1	16	
2	18	
3	16	
4	17	
5	17	
6	16	
Total	100	

The time allotted for this exam is 90 minutes.

Justify your answers carefully. No partial credit can be given for unsubstantiated answers.

If more space is needed, use the back of the previous page and make note of this.

Please write neatly. Answers which are deemed illegible by the grader will not receive credit.

No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students.

In agreeing to take this exam, you are implicitly accepting Harvard University's Honor Code.

(1) Find all critical points of the function

$$f(x, y) = 8x^3 + 6xy - 3y^2 - 24x - 6y + 5$$

Classify each critical point as a local minimum, a local maximum, or a saddle point.

$\nabla F = \langle 24x^2 + 6y - 24, 6x - 6y - 6 \rangle$

$\nabla F = \vec{0}$ when $24x^2 + 6y - 24 = 0$ and $6x - 6y - 6 = 0$

or $x - y = 1$

so $x = 1 + y$

then sub into $24x^2 + 6y - 24 = 0$

$24(1+y)^2 + 6y - 24 = 24 + 48y + 24y^2 + 6y - 24$

$= 0$

or $24y^2 + 54y = 0$ $\quad y(24y + 54) = 0$

$\Rightarrow y = 0$ or $\frac{-54}{24} = -\frac{9}{4}$

then if $y = 0$ $x = 1 + y = 1$

if $y = -\frac{9}{4}$ $x = 1 + y = -\frac{5}{4}$

"D" $= f_{xx} \cdot f_{yy} - (f_{xy})^2 = (48x)(-6) - (6)^2$

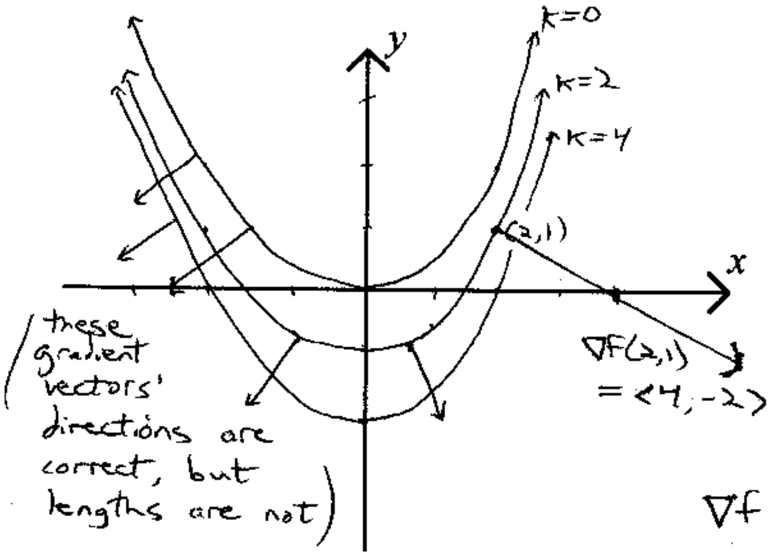
for $(1, 0)$ $D < 0 \Rightarrow$ saddlepoint

for $(-\frac{5}{4}, -\frac{9}{4})$ $D > 0$ and $f_{xx} < 0 \Rightarrow$ local max.

(2) Let $f(x, y) = x^2 - 2y$.

a) Draw several contours (level curves) for this function, and label these curves with the values corresponding to each curve. In particular, draw the contour passing through the point (2, 1).

b) Calculate the gradient ∇f. Illustrate on your level curve diagram the direction of the gradient vector at various points. In particular, find $\nabla f(2, 1)$ and show its direction on the appropriate level curve.



$f(x,y) = k = x^2 - 2y$

or $2y = x^2 - k$

$y = \frac{1}{2}x^2 - \frac{k}{2}$

$\Rightarrow$ level curves are parabolic

for $(2,1)$ $2^2 - 2\cdot 1 = 2$, level $= 2$, $y = \frac{1}{2}x^2 - 1$

$\nabla f = \langle 2x, -2\rangle$

at $(2,1)$ $\nabla F(2,1) = \langle 4, -2\rangle$ as shown on graph

c) Find the directional derivative of f at the point (2, 1) in the direction given by the vector $\mathbf{v} = (-2, 3)$. Are the values of the given function increasing or decreasing in this direction?

unit vector $\vec{u}$ in direction of $\vec{v} \Rightarrow \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -2, 3\rangle}{\sqrt{13}} = \langle \frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\rangle$

then directional derivative is

$\nabla F(2,1)\cdot\vec{u} = \langle 4,-2\rangle\cdot\langle\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\rangle$

$= \frac{-8-6}{\sqrt{13}} = \frac{-14}{\sqrt{13}}$

so values are decreasing in this direction

d) Suppose you move along the curve given parametrically by:

$x(t) = 2e^{2t}$, $y(t) = 2t^3 + 6t + 1$.

What will $\frac{df}{dt}$ be as you travel along this path when $t = 0$?

$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$= 2x\cdot(4e^{2t}) + (-2)(6t^2+6)$

where $x = 2e^{2t}$,

so

$= 2(2e^{2t})(4e^{2t}) - 12t^2 - 12$

when $t=0$, $= 16 - 12 = 4$

(3) Let $f(x,y)=\sqrt{x^4+2xy^2}$.

(a) What is the (best) linear approximation $L(x, y)$ of this function at the point (1, 2)?

(b) Use the approximation in (a) to estimate $f(0.9, 2.03)$.

best linear approx. $\Rightarrow$ tangent plane.

at $(1,2)$ $f(1,2)=\sqrt{1+8}=3$,

tangent plane: at $(1,2)$ $\quad z-3=f_x(1,2)(x-1)+f_y(1,2)(y-2)$

$$f_x=\frac{1}{2}(x^4+2xy^2)^{-1/2}\cdot(4x^3+2y^2)$$

$$\text{so } f_x(1,2)=\frac{1}{2}(9)^{-1/2}\cdot(4+8)=\frac{6}{3}=2$$

$$f_y=\frac{1}{2}(x^4+2xy^2)^{-1/2}\cdot(4xy)$$

$$\text{so } f_y(1,2)=\frac{1}{2}(9)^{-1/2}\cdot 8)=\frac{4}{3}.$$

so Tangent plane is

$$z-3=2(x-1)+\frac{4}{3}(y-2)$$

$$\left(\text{or } z=2x+\frac{4}{3}y-\frac{5}{3}\right)$$

when $x=.9$, $y=2.03$ on tangent plane

$$z=2(.9-1)+\frac{4}{3}(2.03-2)+3$$

$$=2(-.1)+\frac{4}{3}(.03)+3=-.2+.04+3=2.84$$

so $f(.9,2.03)\approx 2.84$

(4) Find the point on the plane $2x + 2y - z = 15$ closest to the point $(-1,1,3)$ using the Method of Lagrange Multipliers.

constraint equation: $2x+2y-z=15$

(so $g(x,y,z) = 2x+2y-z$)

distance formula: distance from a point (x,y,z) on plane to $(-1,1,3)$ is

$$\left((x-(-1))^2+(y-1)^2+(z-3)^2\right)^{1/2}$$

minor trick $\rightarrow$ minimize distance squared $= (x+1)^2+(y-1)^2+(z-3)^2$
$= f(x,y,z)$

so here we go...

$$\nabla g = \langle 2,2,-1\rangle = \lambda \nabla f = \lambda \langle 2(x+1), 2(y-1), 2(z-3)\rangle$$

or
$$2 = 2\lambda(x+1)$$
$$2 = 2\lambda(y-1)$$
$$-1 = 2\lambda(z-3)$$

and $2x+2y-z=15$

$\lambda \neq 0$ ($\Rightarrow$ none of 1st 3 eqns could be solved),

can solve for λ in each: $\lambda = \dfrac{1}{x+1}$ $\quad \lambda = \dfrac{1}{y-1}$ $\quad \lambda = \dfrac{-1}{2(z-3)}$

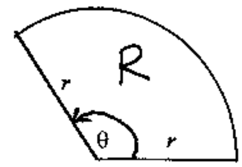
so $x+1 = y-1 = -2(z-3)$, or $x = -2(z-3)-1$
$y = -2(z-3)+1$

now in 4th eqn. $2x+2y-z = 2(-2(z-3)-1)+2(-2(z-3)+1)-z$
$= -8z+24-z = 15$
$\Rightarrow 9 = 9z, \; z=1,$

then $x = -2(z-3)-1 = 3, \quad y = -2(z-3)+1 = 5$

point is $(3,5,1)$, (distance to $(-1,1,3)$ is
$\sqrt{(3-(-1))^2+(5-1)^2+(1-3)^2} = 6$)

(5) A region R is given by a piece of a disk of radius r as shown. If the total perimeter of R is equal to 120 centimeters, what values of r and θ will maximize the area of R? What is this maximum area?



perimeter $= r + r + \underbrace{\text{arclength of sector}}_{= r\theta}$ (as long as θ is in radians!)

$= r(2+\theta)$

area $= \pi r^2 \cdot \dfrac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$

so constraint equation is $r(2+\theta) = 120$,

maximize function $f(r,\theta) = \frac{1}{2} r^2 \theta$

Lagrange! $g(r,\theta) = r(2+\theta)$

$\nabla g = \langle 2+\theta, r \rangle = \lambda \nabla f = \lambda \langle r\theta, \frac{1}{2} r^2 \rangle$

so ① $2 + \theta = \lambda r \theta$

② $r = \frac{1}{2} \lambda r^2$ $\quad \lambda = 0 \Rightarrow r = 0$, no good,

$\lambda, r \neq 0 \Rightarrow \dfrac{r}{r^2} = \frac{1}{2}\lambda$, $\dfrac{2}{r} = \lambda$, from ②

so in ① $2 + \theta = \left(\frac{2}{r}\right) r\theta = 2\theta$,

$2 + \theta = 2\theta \Rightarrow \theta = 2$,

so from constraint equation $r(2+\theta) = 120$

then $r(4) = 120 \Rightarrow r = 30$

so area of 

$= \frac{1}{2} r^2 \theta = \frac{1}{2} 900 \cdot 2$

$= 900$ square centimeters

(max? → of course, check nearby values of r, θ, such as $\theta = 1$, $r = \frac{120}{3} = 40$

area $= \frac{1}{2} r^2 \theta = \frac{1}{2} 1600 \cdot 1 = 800 < 900$)

(6) [Regular and BioChem sections]
Consider the integral

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\, dy\, dx \ .$$

(a) Sketch the region of integration.

(b) Evaluate the integral.

(6) [Physics sections] Compute the line integral $\int_\gamma \mathbf{F}\bullet\mathbf{dx}$ in the following cases:

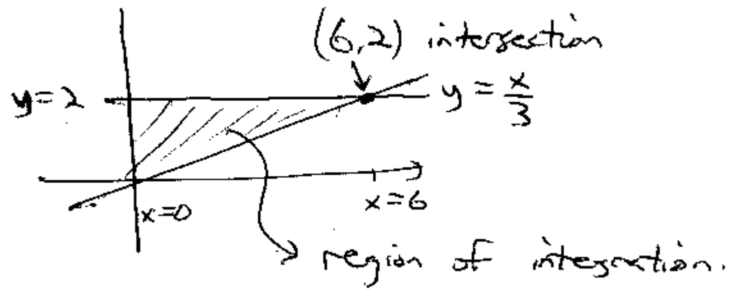
(a) γ is the line segment in $\mathbf{R}^2$ from (0, 1) to (3, 5), and $\mathbf{F}(x, y) = (x^2 - y, xy + 1)$

(b) γ is the path in $\mathbf{R}^3$ from (1, 0, 0) to (1, 0, 2π) along the helix given parametrically by
$\mathbf{x}(t) = (\cos t, \sin t, t)$, and
$\mathbf{F}(x, y, z) = (2x + 2y - 3z, 2x + z + 4, -2z - 3x + y)$

not covered on Spring 2002 midterm #2

Ans. (a) so from limits of integration $\frac{x}{3} \le y \le 2$

while $0 \le x \le 6$



Suspect we're being set up for the classic — "you can't integrate with this order of integration — have to reverse order!"
(This is also pretty obvious → try finding $\int \sqrt{y^3+1}\,dy$! bleah!)

so $\frac{x}{3} \le y \le 2$ while $0 \le x \le 6$ becomes $0 \le x \le 3y$ ← $y = \frac{x}{3}$ line becomes $3y = x$

while $0 \le y \le 2$

integral becomes $\displaystyle\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\, dx\, dy = \int_0^2 \left(\frac{x^2}{2}\sqrt{y^3+1} \Big|_{x=0}^{x=3y} \right) dy$

$\displaystyle = \int_0^2 \frac{9y^2}{2}\sqrt{y^3+1}\, dy$ aha! now can sub

$u = y^3+1 \quad du = 3y^2 dy$

$\frac{1}{3}du = y^2 dy$

$\displaystyle = \frac{9}{2}\int_{y=0}^{y=2} u^{1/2}\cdot\frac{1}{3}du = \frac{9}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}(y^3+1)^{3/2}\Big|_0^2$

$= 9^{3/2} - 1 = 26$