Problem 1) TF questions (20 points)

T F At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \geq 0$.

False. At a local maximum, $f_{yy} \leq 0$.

T F If R is the region bounded by $x^2 + 4y^2 = 1$ then $\int\int_R xy^4 \, dxdy < 0$.

False. The integral is zero because the integral on $R \cap \{x > 0\}$ is the negative of $R \cap \{x < 0\}$.

T F The gradient $\langle 2x, 2y \rangle$ is perpendicular to the surface $z = x^2 + y^2$.

False. The surface is the graph of a function $f(x, y)$. While the gradient of f is perpendicular to the level curve of f, it is only the projection of the gradient to the function $g(x, y, z) = f(x, y) - z$. The later is perpendicular to the surface.

T F The equation $f(x, y) = k$ implicitly defines x as a function of y and $\frac{dx}{dy} = \frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}$.

False. Almost right, the sign is wrong.

T F $f(x, y) = \sqrt{(16 - x^2 - y^2)}$ has both an absolute maximum and an absolute minimum on its domain of definition.

True. The domain of definition is the disc $x^2 + y^2 \leq 16$. The maximum 4 is in the center the absolute minimum 0 at the boundary.

T F If (x_0, y_0) is a critical point of $f(x, y)$ under the constraint $g(x, y) = 0$, and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.

False. The point (x_0, y_0) does not need to be a critical point of f at all.

T F The vector $r_u(u, v)$ of a parameterized surface $(u, v) \mapsto r(u, v) = (x(u, v), y(u, v), z(u, v))$ is normal to the surface.

False. The vector is always tangent to the surface.

T F The identity $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dydx = \int_0^1 \int_0^{\pi/2} r^2 \, d\theta dr$ holds.

False. The area element $d\theta dr$ should be replaced by $r d\theta dr$. So, the right hand side should be $\int_0^1 \int_0^{\pi/2} r^3 \, d\theta dr$.

T F $f(x, y)$ and $g(x, y) = f(x^2, y^2)$ have the same critical points.

False. The function g has always $(0, 0)$ as a critical point, even if f has not.

T F If $f(x, t)$ satisfies the Laplace equation $f_{xx} + f_{tt} = 0$ and simultaneously the wave equation $f_{xx} = f_{tt}$, then $f(x, t) = ax + bt + c$.

False. Take $f(x, t) = xt$. (Here is how we get the general solution: From the two equations, we get $f_{xx} = 0$ and $f_{tt} = 0$. From $f_{xx} = 0$, we obtain that $f(x, t) = a(t)x + c(t)$. From $f_{tt} = 0$, we obtain $a(t)$ and $c(t)$ are linear in t. Therefore the general solution is $f(x, t) = atx + bt + cx + e$).

T F Every smooth function satisfies the partial differential equation $f_{xxyy} = f_{xyxy}$.

True. This follows from Clairot's theorem.

T F The function $f(x, y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0, 0)$.

True. The function is both smaller and bigger than $f(0, 0)$ for points near $(0, 0)$.

T F $\int_0^1 \int_0^{\pi/2} r \, d\theta dr = \pi/4$.

True. This is the area of a quarter of the unit disc.

T F At a saddle point, the directional derivative is zero for two different vectors u, v.

True. The directional derivative can be both positive and negative at a saddle point. By the intermediate value theorem, there are two directions, where the directional derivative vanishes.

T F It is possible to find a function of two variables which has no maximum and no minimum.

True. There are many linear functions like that.

T F The value of the function $f(x, y) = e^x y$ at $(0.001, -0.001)$ can by linear approximation be estimated as -0.001.

True. Because the gradient at $(0, 0)$ is $(0, 1)$ and $f(0, 0) = 0$, the linear approximation is $L(x, y) = y$.

T F For any function $f(x, y, z)$ and any unit vectors u, v, one has the identity $D_{u \times v} f(x, y, z) = D_u f(x, y, z) D_v f(x, y, z)$.

False. The directional derivative in the $u \times v$ direction has nothing to do with the directional derivatives into the other directions. An example, $u = (1, 0, 0), v = (0, 1, 0)$, $f(x, y, z) = x + y$ is an example, where $D_{u \times v} f(x, y, z) = 0$ but $D_u f = 1, D_v f = 1$.

T F Given 2 arbitrary points in the plane, there is a function $f(x, y)$ which has these points as critical points and no other critical points.

True Connect the two points with a line and take this height as the x-axes, centered at the midpoint and with units such that the two points have coordinates (-1,0),(1,0). The function $f(x, y) = -y^2(x^3 - 1)$ has the two points as critical points. One is a local max, the other is a saddle point.

T F The maximum of $f(x, y)$ under the constraint $g(x, y) = 0$ is the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

False This can not be true, because the first problem is the same if we replace $g(x, y)$ with $2g(x, y)$, but this will change the value of the maximum of g on the right hand side.

T F Assume (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xx} f_{yy} - f_{xy}^2 \neq 0$ at this point. Let T be the tangent plane of the surface $S = \{f(x, y) - z = 0\}$ at $P = (x_0, y_0, f(x_0, y_0))$. If the intersection of T with S is a single point, then (x_0, y_0) is a local max or local min.

True. The other possibility would be a saddle point, in which case, the tangent space intersects the surface in two curves which pass through the critical point.

The key is

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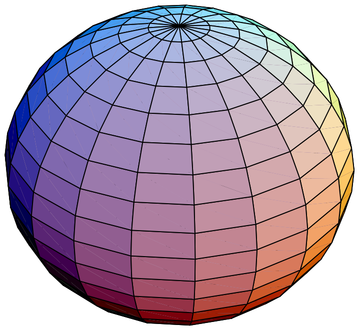
Problem 2) (10 points)

Match the parametric surfaces $S = \vec{r}(R)$ with the corresponding surface integral $\int\int_S \ dS = \int\int_R |\vec{r}_u \times \vec{r}_v| \ dudv$. No justifications are needed.



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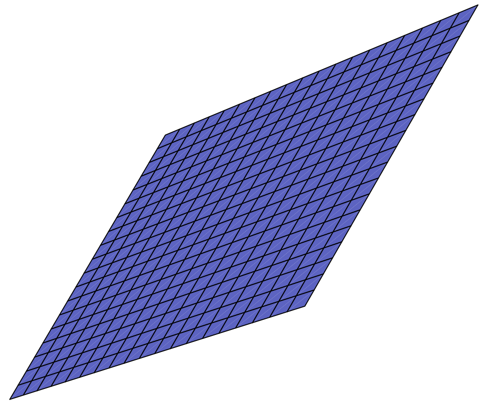

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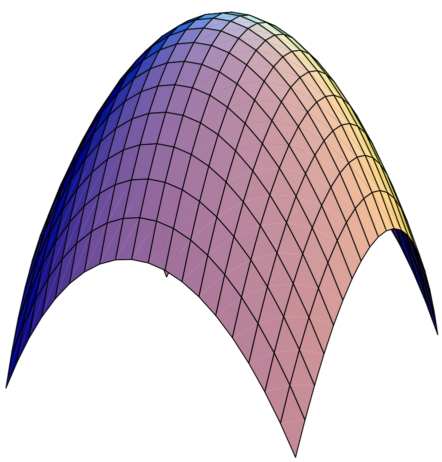

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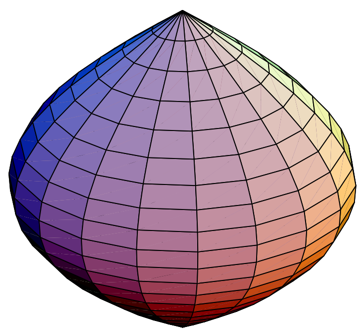

IV

Enter I,II,III,IV here	Surface integral
III	$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \ dudv = \int_0^1 \int_0^1 \sqrt{1+4u^2+4v^2} \ dvdu$
II	$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \ dudv = \int_0^1 \int_0^1 \sqrt{3} \ dvdu$
IV	$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \ dudv = \int_0^{2\pi} \int_0^{\pi} \sin(v)\sqrt{1+\cos(v)^2} \ dvdu$
I	$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \ dudv = \int_0^{2\pi} \int_0^{\pi} \sin(v) \ dvdu$

Problem 3) (10 points)

Find all the critical points of the function $f(x,y) = xy(4-x^2-y^2)$. Are they maxima, minima or saddle points?

Solution. Taking derivatives of $f(x,y) = 4xy - x^3y - xy^3$ gives $\nabla f(x,y) = (4y - 3x^2y - y^3, 4x - x^3 - 3xy^2)$. To solve the system

$$y(4 - 3x^2 - y^2) \quad = \quad 0 \tag{1}$$

$$x(4 - x^2 - 3y^2) = 0 \quad (2)$$

We have the four following possibilites:

1) $y = 0, x = 0$
2) $4 - 3x^2 - y^2 = 0, x = 0$
3) $4 - x^2 - 3y^2 = 0, y = 0$
4) $4 - 3x^2 - y^2 = 0, 4 - x^2 - 3y^2 = 0$.

There are 9 critical points in total

1) gives the critical point $(0, 0)$.
2) gives the critical points $(0, 2), (0, -2)$.
3) gives the critical points $(2, 0), (-2, 0)$.
4) (subtract 3 times the second equation from the first): $(1, 1), (-1, 1), (1, 1), (1, -1)$.

The Hessian determimant (=discriminant) $f_{xx}f_{yy} - f_{xy}^2$ at a general point is $-9(x^4 + y^4) - 16 + 24(x^2 + y^2) + 18x^2y^2$ and $f_{xx}(x, y) = -6xy$.

Applying the second derivative test gives

Critical point	(-2, 0)	(-1, -1)	(-1, 1)	(0, 0)	(1, -1)	(1, 1)	(2, 0)	(-2, 0)	(2, 0)
Discriminant	-64	32	32	-16	32	32	-64	-64	-64
f_{xx}	0	-6	6	0	6	-6	0	0	0
Anslysis	saddle	max	min	saddle	min	max	saddle	saddle	saddle

Problem 4) (10 points)

Let $f(x, y) = e^{(x-y)}$ so that $f(\log(2), \log(2)) = 1$. Find the equation for the tangent plane to the graph of f at $(\log(2), \log(2))$ and use it to estimate $f(\log(2) + 0.1, \log(2) + 0.004)$.

Solution. The graph of f is a level curve of the function $g(x, y, z) = z - f(x, y)$. The gradient at the point $(x_0, y_0, f(x_0, y_0) = (\log(2), \log(2), 1)$ is $(a, b, c) = (-1, 1, 1)$, so that the tangent plane has an equation $ax + by + cz = -x + y + z = d$. and the constant d is obtained from $d = -x_0 + y_0 + z_0 = 1$. Therefore

$$-x + y + z - 1 = 0$$

At the point $(\log(2), \log(2), 1)$, the level surface $g = 0$ is close to the level surface $L(x, y, z) = x - y - z + 1 = 0$. If we plug in $x = 0.1, y = 0.04, z = 0$, we get 1.06.

Remark. We could have worked in two dimensions and estimate $f(x_0 + dx, f(y_0 + dx)$ by $f(x_0, y_0) + (1, -1) \cdot (dx, dy) = 1 + dx - dy$ which is for $dx = 0.1, dy = 0.04$ equal to $1 + 0.1 - 0.04 = \boxed{1.06}$.

Problem 5) (10 points)

Find $\int\int\int_R z^2 \, dV$, where R of the solid obtained by intersecting $\{1 \leq x^2+y^2+z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2+y^2\}$.

Solution: since the result for the double cone is just twice the result for the single cone, we work with the region R obtained with the single cone and multiply at the end with 2. In spherical coordinates, the solid R is as $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi)\sin(\phi) \, d\phi d\theta d\rho = (\frac{2^5}{5} - \frac{1^5}{5})2\pi(\frac{-\cos(\phi))}{3}|_0^{\pi/4} = 2\pi(31/5)(1-2^{-3/2}) \, .$$

The result for the double cone is $\boxed{4\pi(31/5)(1-2^{-3/2})}$.

Problem 6) (10 points)

A **can** is a cylinder with a circular base. Its surface area (top, bottom and sides) is 300π cm^2. What is the maximum possible volume of such a can?

Solution. We have the problem to extremize $f(r,h) = \pi r^2 h$ under the constraint $2\pi r^2 + 2\pi rh = 300\pi$. This is equivalent to extremize $f(r,h) = \pi r^2 h$ under the constraint $g(r,h) = r^2 + rh = 150$.

The Lagrange equations are

$$\begin{aligned} 2\pi rh &= \lambda(2r+h) \\ \pi r^2 &= \lambda r \\ r^2 + rh &= 150 \end{aligned}$$

The second equation gives $\pi r = \lambda$. Plugging in λ into the first equation gtives $h = 2r$. From the last equation, we get $r^2 + 2r^2 = 150$ or $r^2 = 50$. Therefore $r = 5\sqrt{2}cm, h = 10\sqrt{2}cm$. The maximal volume is $\pi r^2 h = \boxed{\pi 500\sqrt{2}}$.

Problem 7) (10 points)

Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{xy^5}{x^2+y^2}\, dy\, dx$.

Solution. The integral is taken over the disc intersected with the first quadrant in the plane. In Polar coordiantes $x^2 + y^2 = r, x = r\cos(\theta), y = r\sin(\theta)$ the integral is therefore

$$\int_0^2 \int_0^{\pi/2} \frac{r\cos(\theta) r^5 \sin^5(\theta)}{r^2}\, r d\theta dr = \int_0^2 \int_0^{\pi/2} r^5 \cos(\theta)\sin^5(\theta)\, d\theta dr = 64/36 = 16/9$$

Solution $\boxed{16/9}$.

Problem 8) (10 points)

a) Find the area of the region D enclosed by the lines $x = \pm 2$ and the parabolas $y = 1 + x^2$, $y = -1 - x^2$.

b) Find the integral of $f(x, y) = y^2$ on the same region as in a). (The result can be interpreted as a moment of inertia).

Solution.

a) $\int_{-2}^2 \int_{-1-x^2}^{1+x^2} 1\; dydx = \int_{-2}^2 2 + 2x^2\; dx = 8 + 2x^3/3|_{-2}^2 = 8 + 32/3 = \boxed{56/3}$.

b) $\int_{-2}^2 \int_{-1-x^2}^{1+x^2} y^2\; dydx = \int_{-2}^2 2(1 + x^2)^3/3\; dx = \int_{-2}^2 (2 + 6x^2 + 6x^4 + 2x^6)/3\; dx = \boxed{2216/35}$

Problem 9) (10 points)

Let $T(u, v) = (v\cos(u), 2v\sin(u)) = (x, y)$.

a) Find the image $S = T(R)$ of the rectangle $R = \{0 \le u \le \pi, 0 \le v \le 1\}$ under the map T and find its area using the formula for the change of variables.

b) Write the integral $\int_0^4 \int_{y/2}^{(y/2)+1} \frac{(2x+y)}{2}\, dxdy$ using uv-coordinates with a change of variables, $T(u, v) = (x, y) = (u + v, 2v)$ and evaluate that integral.

Solution.

a) Because $x^2 + y^2/4 = 1$, the image is half ellipse. The Jacobean $\frac{\partial(x,y)}{\partial(u,v)}$ is $2v$. By the change of variables formula, the area of the ellipse is $\int\int_R 2v\; dA = \int_0^\pi \int_0^1 2v\; dvdu = \boxed{\pi}$.

b) The Jacobean of the coordinate change is $\frac{\partial(x,y)}{\partial(u,v)}$ is 2. The integration region R is the image of the rectangle $[0, 2] \times [0, 1]$ under the map T because $y/2 \le x \le y/2 + 1$ means $0 \le x - y/2 \le 1$ or $0 \le u \le$.

Now, $(2x + y)/2 = (2u + 2v + 2v)/2 = u + 2v$. In the new uv-coordinates, the integral is $\int_0^2 \int_0^1 (u + 2v)\; du\; dv = 10$.